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The Isomorphisms of Unitary Groups over Noncommutative Domains*

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The isomorphisms between projective unitary congruence groups are known when the underlying Witt indices are $\geqslant 3$, the underlying spaces are finite dimensional, and the underlying integral domains are commutative [15, 16]. Here we extend these results to noncommutative domains possessing a division ring of quotients and to arbitrary dimensions. Our development allows unitary and symplectic groups to be treated simultaneously, applies to collinear groups, and unifies the known theories over commutative domains and noncommutative fields. We consider the class of collinear (unitary or symplectic) groups having "enough projective transvections," i.e., at least one on each isotropic line (see Sections 2A and 1B). The chief hurdle, as in the commutative case, is to show that in such groups projective transvections are preserved under isomorphism. From this we get a correspondence of isotropic lines to which the Fundamental Theorem of Projective Geometry can be applied. Then it is easy to show that the isomorphism is of the expected form, i.e., induced by an orthogonality-preserving semilinear bijection (reflexive collinear transformation) between the two underlying spaces; in particular, a unitary group is not isomorphic to a symplectic group.

The group-theoretic CDC approach which was widely successful in the commutative isomorphism theory of classical groups fails when the underlying fields or domains are noncommutative; instead we employ new geometric methods of O'Meara to demonstrate preservation of projective transvections. These methods are applied here assuming the underlying Witt index ν is $\geqslant 3$. It is unlikely they can be adapted to the case $\nu \geqslant 1$, and for $\nu = 0$ it is known [6, p. 242] that the unitary groups over Dedekind domains can be finite, so we cannot hope to obtain the usual correspondence of lines. For $\nu \geqslant 2$, the

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situation is more promising. Here, for example, similar arguments work for the congruence groups in finite dimensions $\geqslant 5$ (see Remark 2.15).

In Section 1 we establish notation, outline the basic facts about reflexive spaces over division rings, introduce the collinear groups and study transvections and certain other transformations. In Section 2 we determine the isomorphisms between unitary/symplectic groups having enough projective transvections (Witt indices $\geqslant 3$) and show how to specialize the results to obtain our main theorems. We also discuss their nonprojective analogs. Finally, in Section 3, we show that a unitary or a symplectic congruence group (having enough projective transvections, Witt index $\geqslant 3$) is not isomorphic to a linear congruence group (underlying dimension $\geqslant 5$).

1. Basic Concepts and Notation

Our notation is based on [11], to which we refer the reader for an account of vector spaces and semilinear algebra over division rings. We let D be a division ring, F its center. V is a vector space of arbitrary dimension and of either orientation t over D. Thus t = right (left) if V is a right (left) space. Scalars will be written on the left, thus for $\alpha, \beta \in D$ and $x \in V$ we have $\alpha(\beta x) = (\alpha\beta)x$ if t = left and $\alpha(\beta x) = (\beta\alpha)x$ if t = right. V' denotes the dual space of V. Linear (and semilinear) transformations will also be written on the left. However we write mappings of division rings on the right and then $\alpha^{\mu J}$, for example, means $(\alpha^{\mu})^J$. V_1 (with associated t_1, D_1, F_1) will denote a second vector space.

A. *Reflexive Spaces and Reflexive Groups*

A standard reference for the material in Section 1A (in finite dimensions) is [3]. A reflexive form on V is a mapping q: $V \times V \to D$ together with an antiautomorphism J of D such that q is additive in each variable and

(i) $\forall x, y \in V$ and $\alpha, \beta \in D$, $q(\alpha x, \beta y) = \begin{cases} \alpha^J q(x, y)\beta \text{ if } V \text{ right,} \\ \beta q(x, y)\alpha^J \text{ if } V \text{ left.} \end{cases}$

(ii) $\exists \epsilon \in D$ such that $\forall x, y \in V$, $q(x, y)^J = \begin{cases} q(y, x)\epsilon \text{ if } V \text{ right,} \\ \epsilon q(y, x) \text{ if } V \text{ left.} \end{cases}$

(For $\dim V \geqslant 2$, condition (ii) is equivalent to symmetry of orthogonality [3, p. 11].) D is commutative $\Leftrightarrow$ J is an automorphism, in particular if J is the identity. Also ϵ satisfies $\epsilon^J = \epsilon^{-1}$ and, for all $\alpha \in D$, $\epsilon\alpha^{J^2} = \alpha\epsilon$ if t = right ($\alpha^{J^2}\epsilon = \epsilon\alpha$ if t = left). Hence J is an involution $\Leftrightarrow$ $\epsilon \in F$. A vector space V equipped with a reflexive form is called a reflexive space (and then we let q, J, ϵ automatically have the above meaning). Let V be a reflexive space.

The reflexive group $I(V)$ of V is defined by

$$I(V) = \{\sigma \in GL(V)\colon q(\sigma x, \sigma y) = q(x, y)\ \forall x, y \in V\}.$$

For a subspace U of V, we define the subspace $U^* = \{x \in V\colon q(x, U) = 0\}$ and the subspace $\operatorname{rad} U = U \cap U^*$. We say U is regular if $\operatorname{rad} U = 0$, degenerate if $\operatorname{rad} U \neq 0$, and totally degenerate if $\operatorname{rad} U = U$. If V is not totally degenerate, the associated J and ϵ are clearly unique. We say $\perp\{V_\alpha\}_{\alpha\in I}$ (where I is some index set) is an orthogonal splitting of V if $V = \bigoplus\{V_\alpha\}_{\alpha\in I}$ (direct sum) and $q(V_\alpha, V_\beta) = 0$ for $\alpha \neq \beta$. We say U splits V if there exists a subspace W of V such that $V = U \perp W$. Clearly $\operatorname{rad} V$ splits V and the splitting $V = \operatorname{rad} V \perp W$ is called a radical splitting of V; note W is regular (and unique up to isometry). If $V = \sum_{\alpha\in I} V_\alpha$ and $q(V_\alpha, V_\beta) = 0$ for $\alpha \neq \beta$, then $\operatorname{rad} V = \sum_{\alpha\in I} \operatorname{rad} V_\alpha$, V is regular $\Leftrightarrow$ each V_α regular; and V regular $\Rightarrow$ $V = \perp\{V_\alpha\}_{\alpha\in I}$.

For any $\alpha \in D$, we can "scale" q to define a new reflexive form q^α on V by

$$q^\alpha(x, y) = \begin{cases} \alpha q(x, y) \text{ if } V \text{ right,} \\ q(x, y)\alpha \text{ if } V \text{ left.}\end{cases}$$

Then q^α has associated $J_\alpha\colon \gamma \to \alpha\gamma^J\alpha^{-1}$ $(\gamma \in D)$ and $\epsilon_\alpha = \epsilon\alpha^J\alpha^{-1}$ if $t = \text{right}$ $(J_\alpha\colon \gamma \to \alpha^{-1}\gamma^J\alpha$ and $\epsilon_\alpha = \alpha^{-1}\alpha^J\epsilon$ if $t = \text{left})$. The reflexive space thus obtained is denoted V^α. Clearly orthogonality is preserved under scaling, so V^α is regular $\Leftrightarrow$ V is regular. And $I(V^\alpha) = I(V)$.

From now on, let V be a *nonzero regular reflexive space.*

1.1. *Let U, W be subspaces of V and let* $\dim U < \infty$. *Then* $\operatorname{codim} U^* = \dim U$, $U^{**} = U$, *and if* $U^* \subseteq W \subseteq V$ *then* $W^{**} = W$. *In particular, if U is regular, then $V = U \perp U^*$. Thus any finite dimensional regular subspace of V splits V* [8].

By a standard application of Zorn's lemma, any totally degenerate subspace of V is contained in a maximal totally degenerate subspace. It is easy to show [18, p. 118] that all maximal totally degenerate subspaces of V either have the same finite dimension $\nu(V)$ or ν (called the Witt index of V) or they are all infinite dimensional (and here we write $\nu = \infty$).

The form q (or the space V) is said to be trace-valued if

$$\forall x \in V \quad q(x, x) \in \begin{cases} \{\alpha + \alpha^J\epsilon^J : \alpha \in D\} \text{ if } V \text{ right,} \\ \{\alpha + \epsilon^J\alpha^J : \alpha \in D\} \text{ if } V \text{ left.}\end{cases}$$

When D is commutative, q fails to be trace-valued $\Leftrightarrow$ $J = \text{identity}$, $\chi(D) = 2$, and $q(x, x) \neq 0$ for some $x \in V$. Also q is automatically trace-valued if $\chi(D) \neq 2$ [3, p. 20]. The assumption of trace-valuedness in a regular isotropic space V is equivalent to assuming V is spanned by its isotropic vectors [18, Lemma 8.1.6].

Our arguments rely heavily on the latter assumption and so, following Dieudonné, we will assume that *all reflexive spaces considered are trace-valued.*

We say q (or V) is alternating if $q(x, x) = 0$ for all x in V, general hermitian if $J \neq$ identity, skew-hermitian if $J \neq$ identity and $\epsilon = -1$. We say $x \in V$ is isotropic $\Leftrightarrow x \neq 0$ and $q(x, x) = 0$; and V is isotropic $\Leftrightarrow$ it contains at least one isotropic vector. Thus $\nu \geqslant 1 \Leftrightarrow V$ is isotropic. We will work mainly with alternating or isotropic skew-hermitian spaces. The restriction to skew-hermitian rather than general hermitian spaces is convenient and causes no real loss of generality since any regular general hermitian space can be scaled to a skew-hermitian space (e.g. if $t =$ right, scale by $\alpha = \theta^J - \theta\epsilon$ where θ is taken in D so that $\alpha \neq 0$, possible since $J \neq$ identity) and our theorems about the isomorphisms are invariant under scaling of the underlying spaces. If V is alternating, we specialize the notation: thus $I(V)$ becomes $\mathrm{Sp}(V)$, the symplectic group of V; and if V is general hermitian $I(V) = U(V)$, the unitary group of V.

A hyperbolic plane is, by definition a reflexive space having a basis $\{x, y\}$ of isotropic vectors such that $q(x, y) = 1$. A hyperbolic plane H in V is regular and, if V is alternating or skew-hermitian, H contains at least three isotropic lines. Any regular isotropic plane in V is hyperbolic since V is trace-valued. A hyperbolic space is an orthogonal sum of hyperbolic planes. A set of vectors $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ in V is called symplectic if $q(x_i, y_i) = 1$ for $1 \leqslant i \leqslant n$ and all other pairs of vectors in the set are orthogonal. Clearly any symplectic set of vectors spans a hyperbolic space and any hyperbolic space has a symplectic base. A finite dimensional regular alternating space V is hyperbolic, so $\dim V = 2\nu(V)$.

1.2. *Let $\{x_i\}_{1 \leqslant i \leqslant n}$ be a base for a totally degenerate subspace U of V. Then there exist vectors, $y_1, \ldots, y_n$ such that $x_1, \ldots, x_n, y_1, \ldots, y_n$ is a symplectic base for a hyperbolic subspace of V. In particular,* $\dim U \leqslant \frac{1}{2} \dim V$. *It is an easy corollary that any finite dimensional subspace of V is contained in a regular finite dimensional subspace of V.*

1.3. *Suppose $\nu(V) \geqslant n \geqslant 1$ and H is a $2n$-dimensional hyperbolic subspace of V. Then $\nu(H^*) = \nu(V) - n$ (with the usual infinite arithmetic, n finite).*

Proof. By induction on n, we can assume $n = 1$. Clearly $\nu(V) \geqslant \nu(H^*) + 1$ and $\nu(H^*) = \infty \Leftrightarrow \nu(V) = \infty$. So it suffices to assume the vectors $\{x_i\}_{1 \leqslant i \leqslant r}$ span an r-dimensional totally degenerate subspace of V where $2 \leqslant r < \infty$ and to produce an $(r - 1)$-dimensional totally degenerate subspace T of H^*. Since H splits V, we can write $x_i = h_i + u_i$ with $h_i \in H$ and $u_i \in H^*$. If all h_i are zero, put $T = Dx_1 + \cdots + Dx_{n-1}$. Suppose some h_i, say h_1 is nonzero. Fix an isotropic vector h_0 in H which is not collinear with h_1. Now by subtracting appropriate multiples of x_1 from the succeeding x_i we can assume that for $i \geqslant 2$, $h_i = 0$ or $h_i = h_0$. Then it follows that for $i \geqslant 2$, $u_i \neq 0$

(otherwise we would have $q(h_0, h_i) = 0$ and H would be degenerate, contrary to assumption), and thus $u_i = x_i - h_i$ is isotropic. Put $T = Dx_2 + \cdots + Dx_r$. Q.E.D.

And it is clear by dimensions that if A is a totally degenerate n-dimensional subspace of V such that $\nu(V) \geqslant n$, then $\nu(A^*) \geqslant \nu(V) - n$.

1.4. *Suppose* $\dim V \geqslant 3$. *Then each line L in V is the intersection of two hyperbolic planes in V, unless $D = \mathbb{F}_3$, $J =$ identity, $\epsilon = 1$ and L is regular* [3, *p.* 43].

1.5. *If V is isotropic, then each isotropic line in V is the intersection of all maximal totally degenerate subspaces of V containing it.*

Proof. It suffices to show that if Dx_1 and Dx_2 are distinct isotropic lines in V, there exists a maximal totally degenerate subspace T of V containing x_1 but excluding x_2. If $q(x_1, x_2) \neq 0$ this is clear. Suppose $q(x_1, x_2) = 0$. Using 1.2, choose vectors y_1, y_2 in V such that x_1, x_2, y_1, y_2 is a symplectic set of vectors. Let W be a maximal totally degenerate subspace of $(Dx_2 + Dy_2)^*$ containing x_1. Then $Dy_2 + W$ is the required subspace T. Q.E.D.

1.6. *Consider V as an abstract vector space. Suppose* $\dim V \geqslant 2$. *Suppose V is a regular reflexive space under each of two reflexive forms q_1 and q_2. Then q_1 and q_2 determine the same orthogonality relation on V if and only if $q_1 = q_2^\alpha$ for some $\alpha \in D$.*

Proof. If $q_1 = q_2^\alpha$, the result is clear. For the converse, define the semilinear injections $\varphi_i\colon V \to V'$ by $(\varphi_i x)(y) = q_i(x, y)$ for $i = 1, 2$. Then the hypothesis implies $(D(\varphi_1 x))^\top = (D(\varphi_2 x))^\top$ for all $x \in V$ whence $\varphi_1 x = \alpha_x(\varphi_2 x)$ where $\alpha_x \in D$. It is easy to see, using the fact φ preserves linear independence (and $\dim V \geqslant 2$), that α_x is actually a constant α independent of x. Hence $\varphi_1 x = \alpha(\varphi_2 x)$ for all $x \in V$, i.e., $q_1 = q_2^\alpha$. Q.E.D.

The concepts and most of the results developed in Section 1B can be found in [11].

B. *Residual Spaces, Fixed Spaces, and Shearings*

Consider V as an abstract vector space. For $\sigma \in GL(V)$, we define the residual space R by $R = (\sigma - 1_V)V$, the fixed space P by $p = \ker(\sigma - 1_V)$, and the residue $\operatorname{res}\sigma$ by $\operatorname{res}\sigma = \dim R = \operatorname{codim} P$. The subspaces R and P of V are called the spaces of σ. We have $\sigma R = R$, $\sigma P = P$, $\operatorname{res}\sigma = 0 \Leftrightarrow \sigma = 1_V$, σ and σ^{-1} have the same R, P, res. If R is a line, plane, hyperplane of V we refer to it as the residual line, plane, hyperplane of σ. Similarly with the fixed line, etc.

Convention. Whenever a σ in $GL(V)$ is under discussion, the letter R will automatically refer to its residual space, the letter P to its fixed space. In the same way R_i and P_i will be associated with σ_i in $GL(V)$. Results 1.7–1.11 below are well known.

1.7. *Let σ_1 and σ_2 be elements of $GL(V)$ and put $\sigma = \sigma_1\sigma_2$. Then $R \subseteq R_1 + R_2$, $P \supseteq P_1 + P_2$ and* $\operatorname{res} \sigma_1\sigma_2 \leqslant \operatorname{res} \sigma_1 + \operatorname{res} \sigma_2$. *Also* $V = P_1 + P_2 \Rightarrow R = R_1 + R_2$ *and* $R_1 \cap R_2 = 0 \Rightarrow P = P_1 \cap P_2$.

1.8. *Let σ and Σ be elements of $GL(V)$. Then the residual and fixed spaces of $\Sigma\sigma\Sigma^{-1}$ are ΣR and ΣP respectively. In particular* $\operatorname{res} \Sigma\sigma\Sigma^{-1} = \operatorname{res} \sigma$, *and* $\sigma\Sigma = \Sigma\sigma \Rightarrow \Sigma R = R$ *and* $\Sigma P = P$.

1.9. *Let σ_1 and σ_2 be elements of $GL(V)$. Then $R_1 \subseteq P_2$ and $R_2 \subseteq P_2 \Rightarrow \sigma_1\sigma_1 = \sigma_2\sigma_1$. Also $\sigma_1\sigma_2 = \sigma_2\sigma_1 \Rightarrow R_1 \subseteq P_2$ and $R_2 \subseteq P_1$, provided either $R_1 \cap R_2 = 0$ or $V = P_1 + P_2$.*

1.10. *Let $\sigma \in GL(V)$. Let W be a subspace of V. If $R \subseteq W$ or $P \supseteq W$, then $\sigma W = W$. The converse holds if* $\operatorname{res} \sigma = 1$. *And if L is a line in V such that $\sigma L = L$, then $L \subseteq R$ or $L \subseteq P$.*

1.11. *Let $\sigma \in GL(V)$. Then $\sigma^2 = 1_V \Leftrightarrow \sigma|_R = -1_R$.*

We say $\sigma \in GL(V)$ is a shearing if $\operatorname{res} \sigma \leqslant 1$ and we call a shearing a transvection if $R \subseteq P$, a dilation if $R \not\subseteq P$. For any $a \in V$, $\rho \in V'$ such that $\rho a \neq -1$ define the mapping $\tau_{a,\rho}$ by $\tau_{a,\rho}x = x + (\rho x)a$ for all $x \in V$. It is easily seen that $\tau_{a,\rho}$ is a shearing in $GL(V)$ and is a transvection $\Leftrightarrow \rho a = 0$. We have $\tau_{a,\rho} = 1_V \Leftrightarrow a = 0$ or $\rho = 0$, and $\tau_{\lambda a,\rho} = \tau_{a,\lambda\rho}$ for all $\lambda \in D$. If $\tau_{a,\rho} \neq 1_v$, then its residual line is Da, and its fixed hyperplane is $\ker \rho$. If $\tau_{a,\rho}$ and $\tau_{b,\rho}$ are defined, then $\tau_{a,\rho}\tau_{b,\rho} = \tau_{a+b,\rho}$. In particular $\tau^m_{a,\rho} = \tau_{ma,\rho}$ for all $m \geqslant 1$. If $\tau_{a,\rho}$ and $\tau_{a,\varphi}$ are defined, then $\tau_{a,\rho}\tau_{a,\varphi} = \tau_{a,\rho+\varphi}$. And $\sigma\tau_{a,\rho}\sigma^{-1} = \tau_{\sigma a,\rho\sigma^{-1}}$ for all σ in $GL(V)$. It is not hard to show that every shearing in $GL(V)$ is a $\tau_{a,\rho}$ with $\rho a \neq -1$. And if $\tau_{a,\rho}$ and $\tau_{a',\rho'}$ are defined and not equal to 1_V, then $\tau_{a,\rho} = \tau_{a',\rho'} \Leftrightarrow$ there is a $\lambda \in D$ such that $a' = \lambda a$ and $\rho' = \lambda^{-1}\rho$.

1.12. *Let τ_1, τ_2 be shearings in $GL(V)$ and let $\alpha \in F$. Exclude the case* $\dim V = 2$, *τ_2 a dilation. Then $(\alpha 1_V)\tau_1 = \tau_2 \Leftrightarrow \alpha = 1$.*

1.13. *Suppose* $\dim V = 2$, *$\sigma \in GL_2(V)$ and $\sigma^2 = 1_V$. If $\chi(D) = 2$, then σ is a transvection.*

1.14. *Let σ_1, σ_2 be non-identity shearings in $GL(V)$. If σ_1, σ_2 are not both dilations, then $\sigma_1\sigma_2 = \sigma_2\sigma_1 \Leftrightarrow R_1 \subseteq P_2$ and $R_2 \subseteq P_1$. If σ_1, σ_2 are both dilations, then $\sigma_1\sigma_2 = \sigma_2\sigma_1 \Leftrightarrow$ (i) $R_1 \subseteq P_2$ and $R_2 \subseteq P_1$, or (ii) $R_1 = R_2$, $P_1 = P_2$ and* *there exists* $0 \neq x_1 \in R_1 = R_2$ *and* $\alpha_1, \alpha_2 \in D$ *such that* $\sigma_i x = \alpha_i x$ *and* $\alpha_1\alpha_2 = \alpha_2\alpha_1$.

Recall from [11] that a collinear transformation of the abstract space V to the abstract space V_1 is a semilinear bijection of V onto V_1. The collinear transformations of V onto V form the group $\Gamma L(V)$ of collinear transformations of V. The projective space $P(V)$ of V is the set of subspaces of V. A projectivity of V onto V_1 is an inclusion preserving bijection of $P(V)$ onto $P(V_1)$. A projectivity of V is a projectivity of V onto V. Each element σ of $\Gamma L(V)$ induces a projectivity $\bar{\sigma}$ of V via the natural homomorphism P of $\Gamma L(V)$ onto the subgroup $P\Gamma L(V)$ of the group of projectivities of V. The radiations of V are the mappings r_α where $r_\alpha x = \alpha x$ for all $x \in V$ ($\alpha \in D$). They form the subgroup $RL(V)$ of $\Gamma L(V)$. Note $r_\alpha \in GL(V) \Leftrightarrow \alpha \in F$. We denote $r_\alpha\sigma$ by $\alpha\sigma$.

1.15. *Suppose* $\dim V \geqslant 2$ *and* U *is a proper subspace of* V. *Let* k *be an element of* $\Gamma L(V)$. *Then* $kL = L$ *for all lines* L *in* $V \backslash U \Leftrightarrow k \in RL(V)$. *In particular* $\ker P = RL$. *And* $\ker P|_{GL} = RL \cap GL = \{r_\alpha : \alpha \in F\}$.

By a representative of an element Σ in $P\Gamma L(V)$ we mean an element k of $\Gamma L(V)$ for which $\bar{k} = \Sigma$. If $\dim V \geqslant 2$, any two representatives of Σ differ by a radiation.

We say two elements, k_1, k_2 of $\Gamma L(V)$ permute projectively $\Leftrightarrow \bar{k}_1$ and $\bar{k}_2$ permute. Obviously permutability implies projective permutability.

1.16. *Suppose* $\dim V \geqslant 2$. *Let* σ *be an element of* $\Gamma L(V)$ *which satisfies any of the following conditions*:

(1) $\operatorname{res} \sigma < \frac{1}{2} \dim V < \infty$;

(2) $\operatorname{res} \sigma = \frac{1}{2} \dim V < \infty$ *and* $\sigma|_R$ *is not a radiation*;

(3) $\operatorname{res} \sigma < \infty$ *with* $\dim V$ *infinite*;

(4) σ *is a transvection.*

Then if σ *permutes projectively with an element* k *in* $\Gamma L(V)$ *it permutes with* k *itself.*

We say a projectivity k of V is a projective shearing (transvection, dilation) if $k = \bar{\sigma}$ for some shearing (transvection, dilation) σ in $GL(V)$. When $\dim V \geqslant 3$, such a representative shearing σ is unique by 1.12, and its R and P are called the residual and fixed spaces of k. The R, P convention above will be extended to projective shearings. And whenever we speak of a projective shearing $\bar{\sigma}$ (in $\dim V \geqslant 3$) we will automatically assume that σ is the unique representative shearing in $GL(V)$ of $\bar{\sigma}$. Note that we sometimes describe elements of $PGL(V)$ in the form $\bar{\sigma}$ with σ in $GL(V)$, and sometimes in the form σ with σ in $PGL(V)$.

1.17. *Suppose* $\dim V \geqslant 3$. *Let* σ_1 *and* σ_2 *be nontrivial projective transvections in* $PGL(V)$. *Then* $\sigma_1\sigma_2$ *is a projective transvection* $\Leftrightarrow R_1 = R_2$ *or* $P_1 = P_2$.

1.18. *Let* X *be a subgroup of* $P\Gamma L(V)$ *consisting entirely of projective transvections. Then all nontrivial elements of* X *either have the same line or the same hyperplane.*

Now let us return to our general assumption that V is a regular reflexive space. It is easy to see that for any σ in $I(V)$, $q(R, P) = 0$.

1.19. *Suppose* $\sigma \in I(V)$ *and* $\operatorname{res} \sigma < \infty$. *Then* $P = R^*$ *and* $R = P^*$.

Proof. We have $\operatorname{codim} P = \dim R = \operatorname{codim} R^*$ by 1.1. And $P \subseteq R^*$. So $P = R^*$ and by 1.1 again $R = R^{**} = P^*$. Q.E.D.

1.20. *If* σ_1, σ_2 *are elements in* $I(V)$ *of finite residue, then* $q(R_1, R_2) = 0 \Rightarrow \sigma_1\sigma_2 = \sigma_2\sigma_1$. *The converse holds if* $R_1 \cap R_2 = 0$ *or, equivalently,* $V = P_1 + P_2$.

Proof. Apply 1.9 using 1.11. Q.E.D.

1.21. *Suppose* $\chi(D) \neq 2$ *and* $\sigma \in I(V)$ *has finite residue and is an involution. Then* $\sigma = -1_R \perp 1_P$.

Proof. By 1.11, $\sigma|_R = -1_R$, so $R \cap R^* = R \cap P = 0$ and R is regular. Hence $V = R \perp P$ by 1.1 and 1.19 and so $\sigma = -1_R \perp 1_P$. Q.E.D.

If a shearing σ falls in $I(V)$, then $P = R^*$, so σ is a transvection $\Leftrightarrow R$ is isotropic.

For the remainder of Section 1B, we suppose V is an alternating or isotropic skew-hermitian space. For any isotropic vector $a \in V$ and any $\lambda \in D$ satisfying $\lambda^J = \lambda$, define the linear map $\tau_{a,\lambda}$ by $\tau_{a,\lambda}x = x + q(a, x)(\lambda a)$ for all x in V. Then it is easy to check that $\tau_{a,\lambda} \in I(V)$; $\tau_{a,\lambda} = 1_V \Leftrightarrow a = 0$ or $\lambda = 0$; for all $\alpha \in D$, $\tau_{\alpha a,\lambda} = \tau_{a,\alpha\lambda\alpha^J}$ if $t =$ right, and $= \tau_{a,\alpha^J\lambda\alpha}$ if $t =$ left; and if $\tau_{a,\lambda} \neq 1_V$, then $\tau_{a,\lambda}$ is a transvection with residual line Da. Thus for each isotropic line L in V there is a transvection σ in $I(V)$ with $R = L$. It is easy to see, using the $\tau_{a,\rho}$ description, that each nontrivial transvection in $I(V)$ is a $\tau_{a,\lambda}$ with $\lambda^J = \lambda$; in fact if $0 \neq a \in R$, there exists $\lambda \in D$ such that $\sigma = \tau_{a,\lambda}$. If $\tau_{a,\lambda}$ and $\tau_{a,\mu}$ are defined, then $\tau_{a,\lambda}\tau_{a,\mu} = \tau_{a,\lambda+\mu}$, $\tau_{a,\lambda}^m = \tau_{ma,\lambda}$ for all $m \geqslant 1$, and $\sigma\tau_{a,\lambda}\sigma^{-1} = \tau_{\sigma a,\lambda}$ for all $\sigma \in I(V)$. If $\tau_{a,\lambda}$ and $\tau_{a',\lambda'}$ are defined and not equal to 1_V, then $\tau_{a,\lambda} = \tau_{a',\lambda'} \Leftrightarrow$ there exists $\alpha \in D$ such that $a' = \alpha a$ and $\lambda = \alpha\lambda'\alpha^J$ if $t =$ right ($\lambda = \alpha^J\lambda'\alpha$ if $t =$ left). In particular, $\tau_{a,\lambda} = \tau_{a,\lambda'} \Leftrightarrow \lambda = \lambda'$.

1.22. *If* σ_1 *and* σ_2 *are nontrivial transvections in* $I(V)$, *then* $\sigma_1\sigma_2$ *is a transvection* $\Leftrightarrow R_1 = R_2$.

1.23. *If* X *is a subgroup of* $I(V)$ *consisting entirely of transvections, then all nontrivial elements of* X *have the same residual line.*

If Da is a regular line in V (so $q(a, a) = \theta \neq 0$) and $\alpha \in D$ is such that $\theta = \alpha^J \theta \alpha$ if $t =$ right ($\theta = \alpha\theta\alpha^J$ if $t =$ left) then the linear map σ defined by $\sigma x = x$ for all $x \in (Da)^*$, $\sigma a = \alpha a$ is a dilation in $I(V)$ with residual line Da.

1.24. *Let σ_1 and σ_2 be non-identity shearings in $I(V)$. If σ_1 and σ_2 are not both dilations, then $\sigma_1\sigma_2 = \sigma_2\sigma_1 \Leftrightarrow q(R_1, R_2) = 0$. If σ_1 and σ_2 are both dilations, then $\sigma_1\sigma_2 = \sigma_2\sigma_1 \Leftrightarrow$ (i) $q(R_1, R_2) = 0$, or (ii) $R_1 = R_2$ and there exists $0 \neq a \in R_1 = R_2$ and $\alpha_1, \alpha_2 \in D$ such that $\sigma_i x = \alpha_i x$ and $\alpha_1\alpha_2 = \alpha_2\alpha_1$.*

1.25. *Suppose* $\dim V \geqslant 3$. *Then the representative shearing of a projective shearing in $PI(V)$ is actually in $I(V)$.*

Proof. It suffices to show that if σ is a shearing in $GL(V)$ and $\alpha\sigma$ is in $I(V)$ for some $\alpha \in F$, then $\alpha^J\alpha = 1$. Since $\dim V \geqslant 3$, obviously P is not totally degenerate, so there exists x, y in P with $q(x, y) = 1$. Then $1 = q(x, y) = q(\alpha\sigma x, \alpha\sigma y) = q(\alpha x, \alpha y)$. So $\alpha^J\alpha = \alpha\alpha^J = 1$. Q.E.D.

1.26. *Suppose* $\dim V \geqslant 3$. *If σ is a projective shearing in $PI(V)$, then $R = P^*$.*

Proof. Apply 1.19 and 1.25. Q.E.D.

1.26. *Let* $\dim V \geqslant 3$ *and suppose σ_1, σ_2 are nontrivial projective transvections in $PI(V)$. Then $\sigma_1\sigma_2$ is a projective transvection $\Leftrightarrow R_1 = R_2$.*

Proof. Apply 1.17 and 1.19. Q.E.D.

1.27. *Let X be a subgroup of $PI(V)$ that consists entirely of projective transvections. Then all nontrivial elements of X have the same residual lines.*

Proof. Apply 1.26. Q.E.D.

C. *Reflexive Collinear Transformations*

In Section 1C, V and V_1 are arbitrary nonzero regular reflexive spaces.

A collinear transformation k of V onto V_1 (with associated isomorphism/antiisomorphism μ of D onto D_1) is said to be reflexive if there exists a constant m (or m_k) in D, dependent on k, such that for all x, y in V

$$q_1(kx, ky) = \begin{cases} m(q(x, y))^\mu \text{ if } V_1 \text{ right,} \\ (q(x, y))^\mu m \text{ if } V_1 \text{ left.} \end{cases}$$

This constant m is uniquely determined by k since V_1 is not totally degenerate and it is called the multiplier of k.

1.29. *Suppose there exists a reflexive collinear transformation k of V onto V_1 (with associated isomorphism/antiisomorphism μ and multiplier m).*

Then

$$m\alpha^{J\mu} = \alpha^{\mu J_1} m \ \forall \alpha \in D, \qquad \epsilon_1 = \epsilon^{\mu} m^{-1} m^{J}, \qquad \textit{if} \quad V_1 \textit{ right},$$

and

$$\alpha^{J\mu} m = m\alpha^{\mu J_1} \ \forall \alpha \in D, \qquad \epsilon_1 = m^{J_1} m^{-1} \epsilon^{\mu}, \qquad \textit{if} \quad V_1 \textit{ left}.$$

In particular, V is alternating (skew-hermitian) $\Leftrightarrow V_1$ is alternating (skew-hermitian).

Proof. The identities are simple consequences of the defining properties of k. Q.E.D.

Composites and inverses of reflexive collinear transformations are again reflexive collinear with $m_{k_1 k} = m_{k_1} m_k^{\mu_1}$ if im k_1 is a right space $(= m_k^{\mu_1} m_{k_1}$, if im k_1 is a left space) and $m_{k_1} = (m_k^{-1})^{\mu^{-1}}$.

If k: $V \to V_1$ is a reflexive collinear transformation with associated μ and m and if V and V_1 are scaled by α and α_1, then k considered as a map k: $V^{\alpha} \to {V_1}^{\alpha_1}$ is still, of course, collinear with respect to μ and in fact is reflexive with multiplier $\alpha_1 m(\alpha^{-1})^{\mu}$ if $t_1 =$ right $((\alpha^{-1})^{\mu}\, m\alpha_1$ if $t_1 =$ left).

A projective reflexive collinear transformation of V onto V_1 is a projective collinear transformation of V onto V_1 which can be expressed in the form $\bar{k}$ for some reflexive collinear transformation of V onto V_1. Composites and inverses of projective reflexive collinear transformations are again projective reflexive collinear.

1.30. *The following statements are equivalent for a collinear transformation k of V onto V_1.*

(1) *k is reflexive collinear.*

(2) *$\bar{k}$ is projective reflexive collinear.*

(3) $q(x, y) = 0 \Leftrightarrow q_1(kx, ky) = 0 \ \forall x, y \in V$.

(4) $kU^* = (kU)^* \quad \forall U \in P(V)$.

Proof. (1) $\Rightarrow$ (2) by the definitions and (1) follows easily from (2) noting that radiations are reflexive collinear. The implications (3) $\Leftrightarrow$ (4) and (1) $\Rightarrow$ (3) are obvious. To prove (3) $\Rightarrow$ (1) first check that $q_2(kx, ky) = q(x, y)^{\mu}$ for all x, y in V defines a regular reflexive form q_2 on V_1. Then

$$q_1(kx, ky) = 0 \Leftrightarrow q(x, y) = 0 \Leftrightarrow q_2(kx, ky) = 0 \qquad \forall x, y \in V.$$

So q_1 and q_2 determine the same orthogonality relation on V_1. Apply 1.6. Q.E.D.

The reflexive collinear transformations of V (i.e. of V onto V) form a subgroup of $\Gamma L(V)$, denoted $\Gamma I(V)$ and called the reflexive collinear group of V. Of course, if V is skew-hermitian $\Gamma I(V) = \Gamma U(V)$, the unitary collinear group of U. And analogously for alternating spaces. We denote $\Gamma I(V) \cap GL(V)$ by $GI(V)$, the group of reflexive similitudes of V.

Each radiation r_α of V is clearly reflexive collinear with multiplier $\alpha^J\alpha$ if $t =$ right ($\alpha\alpha^J$ if $t =$ left) and is a similitude $\Leftrightarrow \alpha \in F$. We have RL and GI are normal subgroups of ΓI.

Note that all representatives in ΓL of an element of $P\Gamma I$ actually fall in ΓI.

If σ is an element of $GI(V)$, it is in $GL(V)$, so R, P, res σ have already been defined.

1.31. *Let* $\sigma \in GI(V)$. *Then*

(1) $R = P^* \Rightarrow \sigma \in I(V)$ *or* $P = 0$, *and conversely if* res $\sigma < \infty$.

(2) $q(P, P) \neq 0 \Rightarrow \sigma \in I(V)$.

Proof. (1) If $R = P^*$ and $P \neq 0$, take p in P and x in V such that $q(x, p) = 1$. Then $\sigma x - x \in R$, so $q(\sigma x - x, p) = 0$ and $q(\sigma x, p) = q(x, p) = 1$. Now the reflexive identity for $q(\sigma x, \sigma p)$ easily shows $m_\sigma = 1$. So $\sigma \in I(V)$. Conversely, if res $\sigma < \infty$, then $\sigma \in I(V) \Rightarrow R = P^*$ by 1.19, while $P = 0 \Rightarrow$ res $\sigma =$ codim $P =$ dim V and so $R = V = P^*$.

(2) Choose vectors p_1, p_2 in P with $q(p_1, p_2) = 1$. Then $1 = q(p_1, p_2) = q(\sigma p_1, \sigma p_2) = m_\sigma$. So $\sigma \in I(V)$. Q.E.D.

1.32. *Suppose* dim $V \geqslant 3$. *Then every shearing in* $\Gamma I(V)$ *is already in* $I(V)$. *Every projective shearing in* $P\Gamma I(V)$ *is already in* $PI(V)$ *and its representative shearing is in* $I(V)$.

Proof. Suppose σ is a shearing in $\Gamma I(V)$. Since $q(P, P) \neq 0$, $\sigma \in I(V)$ by 1.31. The second assertion follows easily from the first. Q.E.D.

1.33. *Suppose* V *is isotropic and* dim $V \geqslant 3$. *Let* k *be an element of* $\Gamma I(V)$ *which stabilizes all isotropic lines in* V. *Then* k *is in* $RL(V)$.

Proof. This is an easy application of 1.4 and 1.15 (the exceptional case in 1.4 can be handled directly). Q.E.D.

We have the following corollary.

1.34. *If an element* k *of* $\Gamma L(V)$ *stabilizes all the isotropic lines in a subspace* U *of* V *such that in a radical splitting* $U =$ rad $U \perp W$ *of* U, W *is isotropic of dimension* $\geqslant 3$, *then* $k|_U$ *is a radiation.*

Finally, if V is alternating or isotropic skew-hermitian and we take a transvection in usual form $\tau_{a,\lambda}$ in $I(V)$, and if $g: V \to V_1$ is reflexive collinear with associated μ and m, we find

$$g\tau_{a,\lambda}g^{-1} = \begin{cases} \tau_{ga,\lambda\mu_{m^{-1}}} \text{ if } V_1 \text{ is right,} \\ \tau_{ga,m^{-1}\lambda\mu} \text{ if } V_1 \text{ is left.} \end{cases}$$

D. *Hyperbolic Transformations*

In this section, we assume V is alternating or isotropic skew-hermitian (in addition to being nonzero, regular, trace-valued).

We say a transformation k in $\Gamma I(V)$ is hyperbolic if $q(kx, x) = 0$ for all isotropic vectors x in V. Clearly, radiations of V are hyperbolic and if $k \in \Gamma I(V)$ is hyperbolic, so are k^{-1}, rk and kr for all r in $RL(V)$. We say $k \in P\Gamma I(V)$ is a projective hyperbolic transformation if k has a representative in $\Gamma I(V)$ which is hyperbolic (in which case all representatives will be hyperbolic).

1.35. *Suppose* dim $V \geqslant 3$ *if V is skew-hermitian. Let $k \in \Gamma I(V)$ be a hyperbolic transformation with associated automorphism μ and multiplier m. Then either k is a radiation or the following hold.*

(1) *D is commutative, $\mu = J$, and $m^J = m$.*

(2) *$k^2 = m1_V$. In particular, every nontrivial projective hyperbolic transformation is an involution.*

(3) *If k moves the isotropic line $Dx_1 = Fx_1$, there exists a regular 4-dimensional hyperbolic subspace U of V, stabilized by k, with a symplectic base $\{x_1, x_2, y_1, y_2\}$ such that*

$$kx_1 = x_2, \qquad kx_2 = mx_1, \qquad ky_1 = my_2, \qquad ky_2 = y_1.$$

Proof. If $\nu = 1$, it is easy to see by the definition of hyperbolic that k stabilizes all isotropic lines in V, hence k is a radiation by 1.15 and 1.33. Now suppose $\nu \geqslant 2$. Take a totally degenerate plane $Dx_1 + Dx_2$ in V. The equality $q(k(x_1 + \alpha x_2), x_1 + \alpha x_2) = 0$ yields, for all $\alpha \in D$, $q(kx_1, \alpha x_2) + q(\alpha^\mu kx_2, x_1) = 0$, and we deduce, considering the case $\alpha = 1$, that for all α in D,

$$\begin{aligned} &q(kx_2, x_1)\alpha = \alpha^{\mu J} q(kx_2, x_1) \text{ if } V \text{ right,} \\ \text{and} \qquad & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (*) \\ &\alpha q(kx_2, x_1) = q(kx_2, x_1)\alpha^{\mu J} \text{ if } V \text{ left.} \end{aligned}$$

First, suppose $q(kx_2, x_1) = 0$ for all pairs of orthogonal isotropic vectors x_1, x_2 in V. Let W be any maximal totally degenerate subspace of V. Then $kW + W$ is totally degenerate by supposition. Hence $kW \subseteq W$ by maximality, and in fact $kW = W$ by considering k^{-1}. It is now an easy consequence of 1.5 that k stabilizes all isotropic lines in V, so k is a radiation by 1.33.

Second, suppose $q(kx_2, x_1) \neq 0$ for some pair of orthogonal isotropic vectors x_1, x_2 in V. Without loss of generality, assume $q(kx_2, x_1) = 1$. Then $(*)$ shows $\alpha = \alpha^{\mu J}$ for all α in D, hence $J = \mu^{-1}$ and J is actually an automorphism, so D $(=F)$ is commutative, and J is an involution, thus $\mu = J$. And $m^J = m$ by 1.29. (Here $\epsilon = -1$). This establishes (1). To prove (2) take arbitrary isotropic lines L_1, L_2 in V. It is easy to choose nonzero vectors x_1 in L_1 and x_2 in L_2

such that $x_1 + x_2$ is isotropic. The equality $q(k(x_1 + x_2), x_1 + x_2) = 0$ implies that $q(kx_1, x_2)^J = q(x_1, kx_2)$ and so $q([k^2 - m1_V]x_1, kx_2) = 0$ (using the fact $\mu = J$). It follows easily that $(k^2 - m1_V)x_1 = 0$, hence the elements k^2 and $m1_V$ of $\Gamma I(V)$ agree on isotropic vectors, and so by 1.33, they differ by a radiation which clearly must be the identity. So $k^2 = m1_V$ and (2) is established. For (3), if x_1 is isotropic and $x_2 = kx_1$ and $Fx_1 \neq Fx_2$, there exists an isotropic vector y_1 in $(Fx_2)^*$ with $q(x_1, y_1) = 1$. Put $y_2 = (1/m)\,ky_1$ and $U = Fx_1 + Fx_2 + Fy_1 + Fy_2$. Q.E.D.

1.36. *Suppose V is alternating and $\sigma \in \mathrm{Sp}(V)$ is a hyperbolic transformation of finite residue. Then* res σ *is even.*

Proof. If σ is a radiation, then $\sigma = \pm 1_V$, and res $\sigma = 0$ or res $\sigma = \dim V$, which is even. Otherwise σ moves an (isotropic) line Dx_1 and σ stabilizes a regular subspace U of V as in part (3) of 1.35. It is easy to check res $\sigma\,|_U = 2$ (since $m = 1$). Now apply induction. Q.E.D.

1.37. *Suppose $k \in \Gamma I(V)$ is a shearing or a hyperbolic transformation and k stabilizes the isotropic line L. Then k permutes with all transvections in $I(V)$ whose residual line is L.*

Proof. Let $L = Da$. If k is a shearing, then $ka = a$ and 1.20 applies. Now suppose k is a hyperbolic transformation with associated automorphism μ and multiplier m. Recall $\mu = J$ and $D = F$. Let $ka = \theta a$ with $\theta \in F$ and suppose $\sigma = \tau_{a,\lambda}$ is a typical transvection in usual form in $I(V)$ with $R = L$. Since $k^2 a = ma$ we get $\theta^J\theta = m$. Compute $k\tau_{a,\lambda}k^{-1} = \tau_{ka,m^{-1}\lambda^\mu} = \tau_{\theta a,m^{-1}\lambda} = \tau_{a,\theta^J\theta m^{-1}\lambda} = \tau_{a,\lambda}$. Q.E.D.

1.38. *Suppose V is alternating and $\chi(F) \neq 2$. Let τ_L be a transvection in* $\mathrm{Sp}(V)$ *with residual line L and let σ_1, σ_2 be hyperbolic transformations in* $\mathrm{Sp}(V)$ *of residue* 2. *Then*

(i) $\sigma_1\sigma_2 = \sigma_2\sigma_1 \Leftrightarrow$ *either* $R_1 = R_2$, *or* $q(R_1, R_2) = 0$ *and* $R_1 \cap R_2 = 0$.

(ii) $\sigma_1\tau_L = \tau_L\sigma_1 \Leftrightarrow L \subseteq R_1$ *or* $L \subseteq P_1$.

Proof. We have $\sigma_i^2 = 1_V$ by 1.34, so $\sigma_i = -1_{R_i} \perp 1_{P_i}$ by 1.21. (i) Suppose $\sigma_1\sigma_2 = \sigma_2\sigma_1$ and $R_1 \neq R_2$. If $R_1 \cap R_2 = 0$ then $q(R_1, R_2) = 0$ by 1.20. If $R_1 \cap R_2$ is a line, Fx say, choose $y \in V$ such that $R_1 = Fx + Fy$ and $q(x, y) = 1$ possible since R_1 is a regular plane) and choose $z \in (Fy)^* \cap R_2$ such that $q(x, z) = 1$ (why possible?). Thus $R_2 = Fx + Fz$. Now $\sigma_1 R_2 = R_2$ and $\sigma_2 R_1 = R_1$ by 1.8, and $y - z \in (R_1 + R_2)^* = P_1 \cap P_2$; so we can compute $y - z = \sigma_1\sigma_2(y - z) = \sigma_1(y + \alpha x + z) = -y - \alpha x + z + \beta x$ for some α, β in F. This is impossible since x, y, z are linearly independent and $\chi(F) \neq 2$. Conversely, if $R_1 = R_2$, then $\sigma_1 = \sigma_2$, and if $q(R_1, R_2) = 0$, apply 1.20.

(ii) Suppose $\sigma_1\tau_L = \tau_L\sigma_1$. Then $\sigma_1 L = L$ by 1.8, and so $L \subseteq R_1$ or $L \subseteq P_1$ by 1.10. The converse is easy. Q.E.D.

2. The Isomorphisms of Unitary and Symplectic Groups

Henceforth, we assume that V is a *nonzero regular reflexive space of orientation* t *over a division ring* D, *equipped with an alternating or trace-valued isotropic skew-hermitian form* q *of Witt index* ν. *And* V_1, t_1, D_1, q_1, ν_1 *is a second similar situation. From Section* 2B *on, we assume* $\nu \geqslant 3$ *and* $\nu_1 \geqslant 3$.

A. *Groups with Enough Projective Transvections*

We say that a subgroup Δ of $P\Gamma I(V)$ has enough projective transvections if for each isotropic line L in V, there is a projective transvection in Δ with $R = L$. For example, $PI(V)$ has enough projective transvections. *From now on* Δ *and* Δ_1 *will denote subgroups of* $P\Gamma I(V)$ *and* $P\Gamma I(V_1)$ *which have enough projective transvections. And* Λ *will denote a group isomorphism* $\Lambda\colon \Delta \to \Delta_1$.

We call D the underlying division ring, V the underlying reflexive space, and ν the underlying Witt index of such a Δ. We say Λ preserves the projective transvection σ in Δ (respectively in Δ_1) if $\Lambda\sigma$ (respectively $\Lambda^{-1}\sigma$) is a projective transvection, and Λ preserves projective transvections if it preserves all projective transvections in Δ and in Δ_1. Given an isotropic line L in V, by the abbreviated phrase "take $\bar{\tau}_L$ in Δ," we mean: let τ_L be a transvection with residual line L such that $\bar{\tau}_L$ is in Δ, (the existence of τ_L will be guaranteed by the assumption of enough projective transvections). If $\bar{\tau}_L \in \Delta$, by a pushforward of τ_L or $\bar{\tau}_L$ we mean a representative of $\Lambda\bar{\tau}_L$ in $\Gamma I(V_1)$. Similarly we speak of a pullback for a projective transvection $\bar{\tau}_{L_1}$ in Δ_1.

The following result distinguishes between projective transvections with distinct residual lines.

2.1. *Suppose* $\dim V \geqslant 5$ *and* $\nu \geqslant 2$. *Let* $\bar{\sigma}_1$, $\bar{\sigma}_2$ *be projective transvections in* Δ *with distinct residual lines. Then if* $\bar{k} \in \Delta$ *permutes with both* $\bar{\sigma}_1$ *and* $\bar{\sigma}_2$ *there exists a conjugate of* $\bar{k}$ *in* Δ *which permutes with exactly one of* $\bar{\sigma}_1$, $\bar{\sigma}_2$.

Proof. Note first that by 1.16, for any transvection σ in $I(V)$, $\bar{k}$ permutes with $\bar{\sigma}$ if and only if k permutes with σ. Suppose if possible that the assertion fails. It follows that if k permutes with the conjugate of σ_2 by an element of $I(V)$ projectively in Δ, it must permute with the same conjugate of σ_1. We consider two cases.

(1) Suppose $q(R_1, R_2) = 0$. Since $\nu \geqslant 2$, P_2 is spanned by isotropic lines (consider a radical splitting of P_2 and use 1.3), so there exists an isotropic line $L = Dx$ in $P_2 \backslash P_1$. Take $\bar{\tau}_L \in \Delta$. Now k permutes with $\tau_L\sigma_2\tau_L^{-1}$, so k permutes with $\tau_L\sigma_1\tau_L^{-1}$ and hence stabilizes its residual line $\tau_L R_1$. Since k stabilizes R_1 also, we have $kL \subseteq L + R_1$.

Now let $R_2 = Dy$ and put $K = D(x + y)$. So K is an isotropic line in $P_2 \backslash P_1$. Take $\bar{\tau}_K \in \Delta$. Put $\sigma'_2 = \tau_K \sigma_2 \tau_K^{-1} = \sigma_2$ and $\sigma'_1 = \tau_K \sigma_1 \tau_K^{-1}$. So k permutes with σ'_2, hence with σ'_1, and the proposition fails for σ'_1, σ'_2 (otherwise it would hold for the original σ_1, σ_2 also, contrary to assumption). We have $L \subseteq P'_2 \backslash P'_1$ so by the preceding paragraph $kL \subseteq L + R'_1$. So $kL \subseteq (L + R_1) \cap (L + R'_1) = L$ and we have shown k stabilizes all isotropic lines in $P_2 \backslash P_1$. If Dz is an isotropic line in $P_2 \cap P_1$ we can always select z so that $z + x$ is isotropic and of course in $P_2 \backslash P_1$. Then it is easy to deduce from the fact that k stabilizes $D(z + x)$, $P_2 \cap P_1$, and L that k stabilizes Dz also. So k stabilizes all isotropic lines in P_2 and by 1.33, $k|_{P_2}$ is a radiation. So $\bar{k}$ is a projective transvection with line R_2. By symmetry its line is also R_1. Impossible.

(2) Suppose $q(R_1, R_2) \neq 0$. Here we have the splitting $V = (P_1 \cap P_2) \perp (R_1 + R_2)$. By 1.3, $P_1 \cap P_2$ is isotropic. So, if $R_2 = Dy$ and if x is any isotropic vector in $P_1 \cap P_2$, then the line $L = D(x + y)$ is isotropic, is in $P_2 \backslash P_1$, and is distinct from R_2. We get $kL \subseteq R_1 + L$ just as in Case (1). By considering $\sigma'_2 = \sigma_2$, $\sigma'_1 = \sigma_2 \sigma_1 \sigma_2^{-1}$ we get $kL = L$ much as in Case (1). (Here we use $L \neq R_2$.) Clearly $kR_2 = R_2$ also, so k stabilizes all isotropic lines in $P_2 \backslash P_1$, and now we proceed to show that $k|_{P_2}$ is a radiation and thus obtain a contradiction just as in Case (1). Q.E.D.

2.2. *Suppose* $\dim V \geqslant 5$, $\nu \geqslant 2$ *and* $\dim V_1 \geqslant 5$, $\nu_1 \geqslant 2$. *Let* $\bar{\sigma}, \bar{\sigma}'$ *be projective transvections in* Δ *with distinct residual lines and suppose* $\Lambda\bar{\sigma} = \bar{\sigma}_1$, $\Lambda\bar{\sigma}' = \bar{\sigma}'_1$ *where* σ_1 *and* σ'_1 *are shearings. Then* R_1 *and* R'_1 *are distinct.*

Proof. Suppose not. Then $R_1 = R'_1$ and since $\nu_1 \geqslant 2$ we can choose an isotropic line L_1 in V_1 orthogonal to $R_1 = R'_1$. Take $\bar{\tau}_{L_1} \in \Delta_1$ and let k be a pullback in $\Gamma I(V)$ of $\bar{\tau}_{L_1}$. Then τ_{L_1} permutes with σ_1 and σ'_1 and so clearly $\bar{k}$, $\bar{\sigma}$, $\bar{\sigma}'$ satisfy the hypotheses of 2.1. The conclusion, carried forward to V_1, asserts the existence of a transvection in $I(V)$, which permutes with exactly one of two shearings in $I(V)$ having the same residual line. This is impossible by 1.14. Q.E.D.

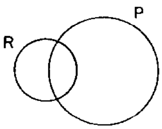
We will make free use of the fact that by 1.16, if the underlying dimension is $\geqslant 5$ we need not distinguish between permuting and projective permuting for two reflexive collinear transformations when at least one of them is linear of residue $\leqslant 2$, in particular is a shearing or a linear hyperbolic transformation of residue $\leqslant 2$.

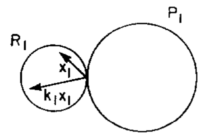
B. *Preservation of Projective Transvections*

Recall henceforth $\nu \geqslant 3$ and $\nu_1 \geqslant 3$.

2.3. *Let* $\bar{\tau}$ *be a projective transvection in* Δ *and suppose* $\Lambda\bar{\tau} = \bar{k}_1$. *Then* $\bar{k}_1$ *is a projective shearing or a projective hyperbolic transformation.*

Proof. Suppose k_1 is not a hyperbolic transformation. So there exists an isotropic vector x_1 in V such that $q_1(k_1x_1, x_1) \neq 0$. Based on any such vector x_1 we construct a "situation" as follows. Let τ_1 be a transvection, projectively in Δ_1, with residual line D_1x_1. Pick $k \in \Gamma I(V)$ such that $\Lambda\bar{k} = \bar{\tau}_1$. Put $\sigma = \tau k \tau^{-1} k^{-1}$ and $\sigma_1 = k_1\tau_1k_1^{-1}\tau_1^{-1}$. So $\Lambda\bar{\sigma} = \bar{\sigma}_1$. We have $R_1 = D_1x_1 + D_1(k_1x_1)$ and R_1 is a hyperbolic plane by choice of x_1; and R is a line or plane in V containing the residual line of τ. Note that $\nu(P_1) \geqslant 2$ since $\nu_1 \geqslant 3$ and P is spanned by isotropic lines since $\nu \geqslant 3$. Call this situation "Sx_1" or "S" and call R_1, P_1 the spaces associated to S. Let $\mathfrak{A}_S$ (or $\mathfrak{A}$) be the set $\{L_1$: L_1 is an isotropic line in P_1 such that there exists a transvection τ_{L_1}, projectively in Δ_1, with residual line L_1 and with a pullback j in $\Gamma I(V)$ which stabilizes R and P and moves an isotropic line in $P\}$.





The main argument is in step (1) below where we show that in a situation as above, if $\mathfrak{A} \neq \varnothing$, then $k_1|_{P_1}$ is a radiation. It follows easily that if we can construct two situations with distinct associated R_1's (and hence distinct P_1's) and for which the $\mathfrak{A}$ sets are nonempty, then k_1 is a radiation on a subspace of codimension 1 in V_1 and we are done. In step (2) we show that all situations S with $\mathfrak{A} = \varnothing$ (if such exist) have the same associated plane R_1. Finally, in step (3) we produce three situations as above with pairwise distinct associated R_1's and the proof is concluded be applying steps (1) and (2).

(1) Suppose $\mathfrak{A} \neq \varnothing$.

(1A) First we show k_1 stabilizes all lines in $\mathfrak{A}$. Let $L_1 \in \mathfrak{A}$. Then we have an isotropic line $L \subseteq P$, and elements $\bar{j} \in \Delta$, $\bar{\tau}_{L_1} \in \Delta_1$ such that $\Lambda\bar{j} = \bar{\tau}_{L_1}$, $jP = P$ and $jL \neq L$. Take $\bar{\tau}_L$ in Δ and let j_1 be a pushforward for $\bar{\tau}_L$. Put $\sigma_2 = j\tau_L j^{-1}\tau_L^{-1}$ and $\sigma_3 = \tau_{L_1} j_1 \tau_L^{-1} j_L^{-1}$. Then σ_2, σ_3 are both linear, $q(R_2, R) = 0$, res $\sigma_2 = 2$, $1 \leqslant$ res $\sigma_3 \leqslant 2$, $L_1 \subseteq R_3 \subseteq P_1$ and $\Lambda\bar{\sigma}_2 = \bar{\sigma}_3$. So τ permutes with σ_2, hence k_1 permutes with σ_3 and so $k_1R_3 = R_3$. If R_3 is a line, then $R_3 = L_1$, and k_1 stabilizes L_1. If R_3 is a plane, we construct in P_1 a plane H_1, stabilized by k_1 and such that $H_1 \cap R_3 = L_1$. Using $\nu \geqslant 3$ choose an isotropic line L_5 in $P_1 \cap L^*{}_1 \backslash R^*{}_3$ (why possible?). Let τ_5 be a transvection with residual line L_5 such that $\bar{\tau}_5$ is in Δ_1. After conjugating σ_3 by τ_5 and pulling things back to V we see that k_1 permutes with $\tau_5\sigma_3\bar{\tau}_5{}^1$, hence k_1 stabilizes $H_1 = \tau_5R_3$. We have thus shown that k_1 stabilizes all lines in $\mathfrak{A}$.

(1B) Here we show that a hyperbolic plane H in P_1 containing a line L in $\mathfrak{A}$ contains at least two other lines in $\mathfrak{A}$. Let H be spanned by the isotropic lines L, K and choose projective transvections $\bar{\tau}_L$, $\bar{\tau}_K$ in Δ_1. By considering a pullback of $\bar{\tau}_K\bar{\tau}_L\bar{\tau}_K^{-1}$ we see that the line $M = \tau_K L$ is in H and in $\mathfrak{A}$. Take $\bar{\tau}_M$ in Δ_1. Then $\tau_M L$ is also in H and in $\mathfrak{A}$ and L, M, $\tau_M L$ are distinct, as required. In particular by step (1A), k_1 stabilizes H.

(1C) Let $L \in \mathfrak{A}$. We show k_1 stabilizes any isotropic line M in P_1 such that $q_1(M, L) \neq 0$. Now M and L span a hyperbolic plane H in P_1 which is stabilized by k_1 by step (1B). By 1.4 we can choose in P_1 a hyperbolic plane $K \neq H$ containing L. Then there exists an isotropic line $N \neq L$ in K and in $\mathfrak{A}$ such that $q_1(M, N) \neq 0$ (otherwise by step (1B) we would have $M \subseteq K^*$ and hence $q_1(M, L) = 0$). N and M span a hyperbolic plane whose intersection with H is M and which is stabilized by k_1 by step (1B), since it contains a line, viz. N, of $\mathfrak{A}$. Hence k_1 stabilizes M.

(1D) Finally we show k_1 stabilizes any isotropic line M in P_1. So fix L in $\mathfrak{A}$. If $q_1(M, L) \neq 0$ apply step (1C). If $q_1(M, L) = 0$, put $L = Dy$, $M = Dz$. Let u be an isotropic vector in P_1 which is nonorthogonal to y and to z (why possible?). Then $Du + Dy$ contains by step (1B) a line N in $\mathfrak{A}$ distinct from L. We have $q_1(N, M) \neq 0$, so by step (1C), k_1 stabilizes M. Hence $k_1|_{P_1}$ is a radiation by 1.33.

(2) Here we must show that if S and S' are two situations with $R_1 \neq R'_1$, we cannot have both $\mathfrak{A} = \varnothing$ and $\mathfrak{A}' = \varnothing$. Suppose $\mathfrak{A}' = \varnothing$. By definition of the set $\mathfrak{A}$ and by 1.33, each projective transvection in Δ_1 whose line is in P_1 has a pullback j in $\Gamma I(V)$ such that $j|_P = 1_P$, hence $j \in I(V)$ by 1.31, and so the residual space of j is contained in R. Hence by standard methods each transvection, projectively in Δ, with residual line in P has a pushforward which permutes with all τ_{L_1} (L_1 isotropic in P_1, $\bar{\tau}_{L_1}$ in Δ_1), thus the pushforward can be taken identity on P_1, and thus linear with residual space in R_1.

Now we claim $P \cap P'$ contains at least two isotropic lines, say L and L'. This is clear if $R \subseteq R'$ or $R' \subseteq R$ or if $\dim \operatorname{rad}(R + R') \geqslant 2$. Otherwise $R + R'$ contains a hyperbolic plane H_0 and we can write $R + R' = H_0 \perp L_0$ where L_0 is a line. Since $P \cap P' = (R + R')^*$, the claim follows using 1.3. Take $\bar{\tau}_L$, $\bar{\tau}_{L'}$ in Δ. By the preceding paragraph, each of these projective transvections has a pushforward with residual space contained in $R_1 \cap R'_1$, a line. This is impossible by 2.2.

(3) Here we must find isotropic vectors x_1, x_2, x_3 in V_1 such that $R_i = Dx_i + D(k_1 x_i)$ are pairwise distinct hyperbolic planes. We already have x_1. If k_1 does not stabilize R_1, putting $x_2 = k_1 x_1$ and $x_3 = k_1 x_2$ suffices. If k_1 stabilizes R_1, and hence P_1, we can choose isotropic vectors y, z in P_1 such that $z \notin Dy + D(k_1 y)$. Now put $x_2 = y$ if $q_1(k_1 y, y) \neq 0$ and $x_2 = x_1 + y$ otherwise; put $x_3 = z$ if $q_1(k_1 z, z) \neq 0$ and $x_3 = x_1 + z$ otherwise. Q.E.D.

2.4. *Suppose Λ and Λ^{-1} carry projective transvections to projective shearings. Then Λ preserves projective transvections.*

Proof. Suppose if possible that for some projective transvection $\bar{\tau}_L$ in Δ, $\Lambda\bar{\tau}_L = \bar{\sigma}_1$ where σ_1 is a dilation in $I(V_1)$. Choose distinct isotropic lines K and M, both $\neq L$, in some totally degenerate plane T in V containing L, and take $\bar{\tau}_K$, $\bar{\tau}_M$ in Δ. By hypothesis we can choose shearings σ_K, σ_M in $I(V_1)$ as pushforwards for these latter two projective transvections. Then σ_1, σ_K, and σ_M have residual lines which are distinct by 2.2 and mutually orthogonal by the usual considerations of permutability. Now since at least one of these three lines is regular, it is easy to see that they are linearly independent. Hence since $\nu_1 \geqslant 3$, we can find an isotropic line L_1 in V_1 orthogonal to two of these lines but not to the third. Now take $\bar{\tau}_{L_1}$ in Δ_1. By hypothesis, we can choose σ in $I(V)$ of residue 1 such that $\bar{\sigma} \in \Delta$ and $\Lambda\bar{\sigma} = \bar{\tau}_{L_1}$. But τ_{L_1} permutes with exactly two of σ_1, σ_K, σ_M, and pulling things back to V via Λ, we find $q(R, T) = 0$, so the pullback σ permutes with all three of τ_L, τ_K, τ_M. Contradiction. Thus Λ preserves projective transvections in Δ and, by symmetry, the result follows. Q.E.D.

2.5. *Suppose Λ preserves one projective transvection. Then Λ preserves all projective transvections.*

Proof. By simple order considerations, it is easy to see that the hypothesis implies the underlying characteristics are equal, and if this common characteristic is not 2, then 2.4 applies (in view of 2.3) and we are done. So suppose $\chi(D) = \chi(D_1) = 2$.

By symmetry, it clearly suffices to show that if $\Lambda\bar{\tau}_L = \bar{\tau}_{L_1}$ for projective transvections $\bar{\tau}_L$ in Δ and $\bar{\tau}_{L_1}$ in Δ_1, and if $\bar{\tau}_K$ is any projective transvection in Δ then $\Lambda\bar{\tau}_K$ is a projective transvection in Δ_1. By suitably conjugating τ_L we can assume $q(L, K) \neq 0$. Let k_1 be a pushforward for τ_K. Then $\bar{\tau}_K\bar{\tau}_L\bar{\tau}_K^{-1}$ and $\bar{k}_1\bar{\tau}_{L_1}\bar{k}_1^{-1}$ are projective transvections with residual lines $M = \tau_K L$ and $M_1 = k_1 L_1$. Clearly the subspaces Π and Π_1 spanned, respectively, by L, M and L_1, M_1 are hyperbolic planes. Let X_1 be a typical isotropic line in $\Pi^*{}_1$ and take $\bar{\tau}_{X_1}$ in Δ_1. By 2.3 there exists k in $\Gamma I(V)$ such that k is a shearing or a hyperbolic transformation, $\bar{k} \in \Delta$, and $\Lambda\bar{k} = \bar{\tau}_{X_1}$. By a standard permuting argument, it is easy to see that k stabilizes K (note that a hyperbolic transformation, by its definition, must stabilize all isotropic lines in any hyperbolic plane which it stabilizes). Hence, k permutes with τ_K by 1.37, and so k_1 permutes with τ_{X_1}, so k_1 stabilizes X_1. Thus we can assume $k_1 |_{\Pi^*{}_1} = 1_{\Pi^*{}_1}$. Since $\chi(D) = 2$, we see that $\bar{\tau}$, hence $\bar{k}_1$, and hence $k_1 |_{\Pi_1}$ are involutions. So by 1.13, $k_1 |_{\Pi_1}$ is a transvection, hence so is k_1. Q.E.D.

2.6. *Suppose $\bar{\tau}$ is a projective transvection in Δ and $\Lambda\bar{\tau} = \bar{k}_1$ is a projective hyperbolic transformation in $P\Gamma I(V_1)$.*

(1) *Exclude the case V_1 skew-hermitian and* $\dim V_1 = 6$. *Then V_1 is alternating and k_1 can be chosen in $I(V_1) = \mathrm{Sp}(V_1)$ of residue* 2.

(2) *Assume V_1 is skew-hermitian and* $\dim V_1 = 6$. *Then V is alternating.*

Proof. By 1.33, k_1 moves an isotropic line L_1 in V_1. Take $\bar{\tau}_{L_1} \in \Delta_1$ with pullback k, say, in $\Gamma I(V)$. Put $\sigma = \tau k \tau^{-1} k^{-1}$ and $\sigma_1 = k_1 \tau_{L_1} k_1^{-1} \tau_{L_1}^{-1}$. We have $\Lambda\bar{\sigma} = \bar{\sigma}_1$, R_1 is a totally degenerate plane containing L_1, and R is a line or plane containing the residual line L, say, of τ.

(1) Here V_1 is alternating if $\dim V_1 = 6$. By standard methods we can find a conjugate σ' to σ in $\Gamma I(V)$ and a conjugate σ'_1 to σ_1 in $\Gamma I(V_1)$ such that

$$\Lambda\overline{\sigma'} = \overline{\sigma_1'}, \qquad R \cap R' = L.$$

(Start by suitably conjugating σ by a transvection in $I(V)$ that is projectively in Δ: let $L = Dx$, so $R = Dx + D(kx)$; if R is a hyperbolic plane, take an isotropic vector y in P and use a transvection with residual line $D(x + y)$; if R is a totally degenerate plane, take an isotropic vector y in V satisfying $q(x, y) = 0$, $q(kx, y) \neq 0$ and use a transvection on Dy.) Now for any isotropic line K_1 in $(R_1 + R'_1)^*$, take $\bar{\tau}_{K_1}$ in Δ_1. By 2.3 we can write $\Lambda^{-1}\bar{\tau}_{K_1} = \bar{j}$ where j is a shearing or a hyperbolic transformation. Clearly j permutes with σ and σ', so j stabilizes $R \cap R' = L$, so j permutes with τ by 1.37. So k_1 permutes with τ_{K_1}, so k_1 stabilizes K_1. Thus k_1 stabilizes all isotropic lines in $(R_1 + R'_1)^*$.

Next we show that k_1 is linear. First, if $R_1 + R'_1$ $(=W)$ is regular, it is easy to see using 1.3 that W^* is regular and isotropic. So by 1.33 when $\dim V_1 \geqslant 7$ and 1.15 when $\dim V_1 = 6$, we can arrange that k_1 is identity on W^*, in particular, k_1 is linear. Second, if W is degenerate, we claim W^* has a totally degenerate plane subspace. Now $\dim W \leqslant 4$. Consider $m = \dim(\operatorname{rad} W)$. If $m \geqslant 2$, done. So suppose $m = 1$. If $\dim W = 4$, let $\operatorname{rad} W = D(z_1 + z'_1)$ where $z_1 \in R_1$, $z'_1 \in R'_1$. Then it is easy to see that z_1, z'_1 also are in $\operatorname{rad} W$, i.e. $\operatorname{rad} W$ is contained in R_1 or in R'_1, say in R_1. Then we can write $W = \operatorname{rad} W \perp T$ where T is regular and contains R'_1, impossible by 1.2. Clearly $\dim W \neq 2$, so $\dim W = 3$. So $W = \operatorname{rad} W \perp T$ where T is a hyperbolic plane and since $\nu_1 \geqslant 3$, it is easy to see that $W^* = (R_1 + R'_1)^*$ has a totally degenerate plane subspace, as claimed. So 1.15 implies that k_1 can be chosen linear.

It follows by 1.35(1) that $J_1 =$ identity, hence V_1 is alternating. So we can assume k_1 is identity on $(R_1 + R'_1)^*$. Then by 1.35(2) k_1 has multiplier 1, so $k_1 \in I(V_1) = \operatorname{Sp}(V_1)$ and $k_1^2 = 1_{V_1}$. Now $\operatorname{res} k_1$ is even by 1.36, and $\leqslant 4$ since the fixed space of k_1 contains $(R_1 + R'_1)^*$. But $\operatorname{res} k_1 = 4$ implies that $R_1 + R'_1$ is the residual space of k_1, which is impossible in view of 1.11, since k_1 moves the line L_1 or R_1. Hence $\operatorname{res} k_1 = 2$.

(2) Here V_1 is skew-hermitian and $\dim V_1 = 6$. By 1.35(3) there exists a four dimensional regular subspace U_1 of V_1 stabilized by k_1 and containing R_1. Let K_1 be any isotropic line in the hyperbolic plane U^*_1. Clearly $k_1 K_1 = K_1$. So k_1 stabilizes the totally degenerate space $R_1 + K_1$ $(=T_1)$. Since k_1 is nonlinear, there is an (isotropic) line L'_1 in $T_1 \backslash R_1$ moved by k_1. Now starting

with a transvection on the line L'_1, form elements $\overline{\sigma'} \in \Delta$, $\overline{\sigma'_1} \in \Delta_1$ just as $\bar{\sigma}$, $\overline{\sigma_1}$ were obtained from the line L_1. We have $R_1 + R'_1 = R_1 + K_1$, so $R_1 \cap R'_1$ is a line, K_3 say. Take $\bar{\tau}_{K_3} \in \Delta_1$ with a pullback j, say, in $\Gamma I(V)$. First, we have $R = R'$ for otherwise, since τ_{L_1} permutes with σ_1 and σ'_1 we would have k permuting with σ and σ', hence stabilizing $R \cap R' = L$, hence permuting with τ, which is not so. Now by standard methods we find j stabilizes L and all isotropic lines in $P = P'$. It follows easily that $\bar{j}$ cannot be a projective dilation. Thus by 2.3 and 2.5, j is a hyperbolic transformation, and j is linear, so V is alternating by 1.35(2). Q.E.D.

2.7. *Λ preserves projective transvections.*

Proof. Suppose not. Then by 2.3, 2.4 and 2.6 it is immediate that at least one of the underlying spaces is alternating. So without loss of generality, let V_1 be alternating. By 2.3, 2.5 and 2.6(1) it is clear that for each projective transvection $\bar{\tau}$ in Δ, $\Lambda\bar{\tau}$ has a representative hyperbolic transformation k_1 in $\mathrm{Sp}(V_1)$ of residue 2. And, since there are no dilations of order two in characteristic two, it is easy to see by 2.3 and 2.5 that either (1) $\chi(F_1) = 2$ and Λ^{-1} carries all projective transvections in Δ_1 to projective hyperbolic transformations in Δ or (2) $\chi(F_1) \neq 2$, V is skew-hermitian, and Λ^{-1} carries all projective transvections in Δ_1 to projective dilations in Δ. In Case (1) we obtain a contradiction by an argument almost identical to that in [10, Sect. 5.2.8]. In Case (2), for each projective transvection $\bar{\tau}_L$ in Δ, we can write by 2.5 and 1.11, $\Lambda\bar{\tau}_L = \bar{\sigma}$ where $\sigma = 1_P \perp -1_R$ and R is a hyperbolic plane in V_1. A moment's reflection shows that corresponding to distinct L's, we have distinct R's. Since Δ has enough projective transvections and card $D \geqslant 3$, we can choose four transvections τ_i $(1 \leqslant i \leqslant 4)$, projectively in Δ, with four distinct residual lines in some totally degenerate plane Π contained in V. Let $\Lambda\bar{\tau}_i = \bar{\sigma}_i$ with $\sigma_i = 1_{P_i} \perp -1_{R_i}$ as above. The τ_i permute in pairs, hence so do the σ_i and since the planes R_i are pairwise distinct, we have $q_1(R_i, R_j) = 0$ and $R_i \cap R_j = 0$ for distinct i, j by 1.38; so we have an orthogonal sum $R_1 \perp R_2 \perp R_3 \perp R_4$ in V_1. Now take an (isotropic) line L_1 in V_1 orthogonal to R_1 and to R_2 but not contained in R_3 or R^*_3 (why possible?) and take $\bar{\tau}_{L_1}$ in Δ_1. By assumption, τ_{L_1} has a pullback which is a dilation, j say. By standard arguments we find τ_{L_1} permutes with σ_3, so $L_1 \subseteq R_3$ or $L_1 \subseteq R^*_3$ by 1.38. Contradiction. Q.E.D.

C. *The Isomorphism Theorems in General*

If g is any projective reflexive collinear transformation of V onto V_1 then the mapping Φ_g of $P\Gamma I(V)$ onto $P\Gamma I(V_1)$ defined by $\Phi_g k = gkg^{-1}$ $(k \in P\Gamma I(V))$ is well known to be an isomorphism of $P\Gamma I(V)$ onto $P\Gamma I(V_1)$. The passage from the preservation of projective transvections established in Section 3A to the existence of a Φ_g inducing Λ is the usual one. Let us outline the details.

First characterize the maximal groups of permuting projective transvections in Δ as the subgroups $\Delta(L)$ where L is any isotropic line in V and $\Delta(L)$ is the set of projective transvections in Δ with residual line L plus the identity. This yields an orthogonality-preserving bijection π of the set of isotropic lines of V onto the set of isotropic lines in V_1 via the relation $\Lambda\Delta(L) = \Delta_1(\pi L)$. By a result of Tits [18, 8.6(II)] (or a direct argument following [16, p. 1012]), the map π can be extended to an orthogonality-preserving projectivity of $P(V)$ onto $P(V_1)$ which by the Fundamental Theorem of Projective Geometry and 1.30 is induced by a reflexive collinear transformation g of V onto V_1. By considering the actions of Φ_g and Λ on projective transvections and their conjugates in Δ it is easy to show $\Lambda = \Phi_g |_\Delta$ and that this equation uniquely defines g. (See [10, 5.3.3].) Thus we obtain our main theorems. (In view of our remarks on scaling in Section 1A, for the following three Theorems we need not restrict our general hermitian spaces to be skew-hermitian.)

2.8. Theorem. *Let V and V_1 be regular trace-valued isotropic general hermitian spaces each of Witt index $\geqslant 3$ over division rings D and D_1. Let Δ and Δ_1 be subgroups of $P\Gamma U(V)$ and $P\Gamma U(V_1)$ respectively which have enough projective transvections. Let Λ be a group isomorphism of Δ onto Δ_1. Then there exists a unique projective unitary collinear transformation g of V onto V_1 such that*

$$\Lambda k = gkg^{-1} \qquad \forall k \in \Delta.$$

2.9. Theorem. *Let V and V_1 be regular alternating spaces, each of Witt index $\geqslant 3$ (equivalently, of dimension $\geqslant 6$) over fields F and F_1. Let Δ and Δ_1 be subgroups of $P\Gamma\mathrm{Sp}(V)$ and $P\Gamma\mathrm{Sp}(V_1)$ respectively, which have enough projective transvections. Let Λ be a group isomorphism of Δ onto Δ_1. Then there exists a unique projective symplectic collinear transformation g of V onto V_1 such that*

$$\Lambda k = gkg^{-1} \qquad \forall k \in \Delta.$$

2.10. Theorem. *Let V be a regular trace-valued isotropic general hermitian space of Witt index $\geqslant 3$ over a division ring D. Let V_1 be a regular alternating space of dimension $\geqslant 6$ over a field F_1. Suppose Δ, Δ_1 are subgroups of $P\Gamma U(V)$, $P\Gamma\mathrm{Sp}(V_1)$ respectively, which have enough projective transvections. Then Δ is not isomorphic to Δ_1.*

Now let us show how to extend these isomorphism theorems to the non-projective case. We say a subgroup Γ of $\Gamma I(V)$ has enough transvections if for each isotropic line L in V there is at least one transvection σ in Γ with $R = L$. Let Γ and Γ_1 denote subgroups of $\Gamma I(V)$ and $\Gamma I(V_1)$ which have enough transvections and let Φ denote a group isomorphism $\Phi: \Gamma \to \Gamma_1$. Note that $\bar{\Gamma} = P\Gamma$ and $\bar{\Gamma}_1 = P\Gamma_1$ are subgroups of $P\Gamma I(V)$ and $P\Gamma I(V_1)$ which have enough projective transvections so that the preceding theory for Δ and Δ_1 applies to $\bar{\Gamma}$ and $\bar{\Gamma}_1$.

We say Φ collapses on a subset X of Γ if $\Phi X \subseteq RL(V_1)$. It is easy to show that Φ collapses on the transvections of $\Gamma \Leftrightarrow \Phi$ collapses on the linear elements of $\Gamma \Leftrightarrow \Phi^{-1}$ collapses on the transvections of Γ_1, and we have the following result (see [11, p. 125] for the linear versions).

2.11. *Φ does not collapse on the transvections of Γ if any one of the following conditions is satisfied.*

(1) *D is commutative*;

(2) $G \subseteq GL(V)$;

(3) *G contains a linear involution* $\neq \pm 1_V$.

2.12. *If Φ does not collapse on the transvections of G then Φ naturally induces an isomorphism $\bar{\Phi}$: $\bar{\Gamma} \to \bar{\Gamma}_1$ by the equation*

$$\bar{\Phi}\bar{k} = \overline{\Phi k} \qquad \forall k \in \Gamma.$$

Proof. It is clearly enough to show that $\Phi(\Gamma \cap RL(V)) = \Gamma_1 \cap RL(V_1)$, in fact, by considering Φ^{-1}, that $\Phi(\Gamma \cap RL(V)) \subseteq \Gamma_1 \cap RL(V_1)$. So suppose for a contradiction that $\Phi(\Gamma \cap RL(V)) \not\subseteq \Gamma_1 \cap RL(V_1)$. Let $\mathcal{O} = \{L_1 : L_1$ is an isotropic line in V_1 and $(\Phi r)L_1 \neq L_1$ for some radiation r in $\Gamma\}$. By our supposition, we see that $\mathcal{O} \neq 0$. Now let σ be a typical linear element in Γ. We will show that $k_1 = \Phi\sigma$ is a radiation. The method of proof is almost identical to that of step (1) of 2.3 to which the reader may refer for details of the following outline. Let $L_1 \in \mathcal{O}$ (say r is the radiation with $(\Phi r)L_1 \neq L_1$ and $\Phi r = j_1$) and take $\tau_{L_1} \in \Gamma_1$. Put $\sigma_1 = j_1 \tau_{L_1} j_1^{-1} \tau_{L_i}^{-1}$. Then R_1 is a regular plane and $\Phi^{-1}\sigma_1$ is a radiation, which permutes with σ by linearity. So k_1 permutes with σ_1 and k_1 stabilizes R_1. Conjugate σ_1 by an appropriate transvection in Γ_1 to obtain σ'_1 with $R_1 \cap R'_1 = L_1$. Then k_1 permutes with σ'_1 also and so k_1 stabilizes L_1. Next show any plane in P_1 that contains a line in $\mathcal{O}$ is spanned by lines in $\mathcal{O}$, by a conjugating argument. Then show k_1 stabilizes any isotropic line orthogonal to L_1 by expressing it as the intersection of two hyperbolic planes spanned by lines in $\mathcal{O}$. Finally show k_1 stabilizes any isotropic line orthogonal to L_1 by exhibiting an intermediate line in $\mathcal{O}$ which is nonorthogonal to each of them. Thus we have shown that Φ collapses on transvections (in fact, linear elements) in Γ, contrary to hypothesis. Q.E.D.

Thus there are obvious analogs to 2.8, 2.9, and 2.10 for nonprojective groups of reflexive collinear transformations which have enough transvections.

D. *The Isomorphisms over Domains*

Here we show that our main results apply to certain so-called congruence groups.

Let $\mathfrak{o}$ be an integral domain with identity possessing a division ring of quotients D. This means that D contains $\mathfrak{o}$ as a subring and each α in D can be expressed in the form $\alpha = ab^{-1}$ with $a, b \in \mathfrak{o}$ and also in the form $\alpha = c^{-1}d$ with c, d in $\mathfrak{o}$. By a fractional right ideal $\mathfrak{a}$ with respect to $\mathfrak{o}$ we mean a nonzero additive subgroup of D such that $\mathfrak{a}p \subseteq \mathfrak{a}$ for all p in $\mathfrak{o}$, and there exists $0 \neq q \in D$ such that $q\mathfrak{a} \subseteq \mathfrak{o}$. A fractional left ideal is defined analogously.

By an $\mathfrak{o}$-module on the vector space V (see [11, p. 128]) we mean an $\mathfrak{o}$-module in V whose D-span is V. A bounded $\mathfrak{o}$-module on V is one which is sandwiched between two free $\mathfrak{o}$-modules on V. Let M be a bounded $\mathfrak{o}$-module on V and let $\mathfrak{a}$ be an integral (i.e. nonzero two-sided) ideal of $\mathfrak{o}$. Then $\mathfrak{a}M = \{\sum_{\text{finite}} \alpha x \colon \alpha \in \mathfrak{a}, x \in M\}$ is a bounded $\mathfrak{o}$-module on V. We have the integral linear group $GL(M) = \{\sigma \in GL(V) \colon \sigma M = M\}$ and the linear congruence group $GL(M; \mathfrak{a}) = \{\sigma \in GL(V) \colon (\sigma - 1_V)M \subseteq \mathfrak{a}M\}$. Clearly $GL(M; \mathfrak{a}) \lhd GL(M)$ and $GL(M; \mathfrak{o}) = GL(M)$. We have the integral reflexive group $I(M) = GL(M) \cap I(V)$ and the reflexive congruence group $I(M; \mathfrak{o}) = GL(M; \mathfrak{a}) \cap I(V)$. Clearly $I(M; \mathfrak{a}) \lhd I(M)$ and $I(M; \mathfrak{o}) = I(M)$. The projective versions of these groups are defined by applying P. If V is scaled by α, denote M as a module on V^α by M^α. Since $I(V^\alpha) = I(V)$, we have $I(M^\alpha, \mathfrak{a}) = I(M, \mathfrak{a})$. Now let us further suppose $\mathfrak{o}^J = \mathfrak{o}$ (e.g. integral quaternions with the usual conjugation). Recall V is alternating or isotropic skew-hermitian. (The next result does not require that ν be $\geqslant 3$.)

2.13. *Let M be a bounded $\mathfrak{o}$-module on V and let $\mathfrak{a}$ be any integral ideal of $\mathfrak{o}$. Then $PI(M, \mathfrak{a})$ has enough projective transvections $\Leftrightarrow$ for each isotropic vector a in M, $q(a, M)$ is contained in a fractional right* (*left*) *ideal with respect to $\mathfrak{o}$ if V is a right* (*left*) *space.*

Proof. Suppose first V is a right space. Since M spans V, a typical transvection in $I(V)$ can be written in usual form $\tau_{a,\lambda}$ with a an isotropic vector in M, and, of course, $\lambda^J = \lambda$. Then it is easy to see that

$$\tau_{a,\lambda} \in I(M, \mathfrak{a}) \Leftrightarrow \lambda q(a, M) \subseteq \mathfrak{a}.$$

Thus the $\Rightarrow$ direction is clear. Conversely, if $a \in M$ is isotropic, then by hypothesis there exists $0 \neq p \in \mathfrak{o}$ such that $pq(a, M) \subseteq \mathfrak{o}$. Take any nonzero element r in $\mathfrak{a}$. Put $\lambda = (rp)^J rp$. Then since $\mathfrak{o}^J = \mathfrak{o}$, $\lambda q(a, M) \subseteq \mathfrak{a}$ and $\tau_{a,\lambda} \in I(M, \mathfrak{a})$. Thus $I(M, \mathfrak{a})$ has enough transvections. Proceed similarly if V is a left space. Q.E.D.

2.14. *Remark.* If V is finite dimensional, or if $q(M, M) \subseteq \mathfrak{o}$, then it is easy to see by 2.13 that $I(M, \mathfrak{a})$ has enough transvections and hence $PI(M, \mathfrak{a})$ has enough projective transvections.

So our main theorems 2.8, 2.9 and 2.10 apply to projective congruence groups having enough projective transvections. And, of course, there are obvious nonprojective analogs.

2.15. *Remark.* By modifying the arguments in Section 2B we can relax the Witt index assumptions to $\nu \geqslant 2$, $\nu_1 \geqslant 2$ in the case of isomorphic projective reflexive congruence groups Δ and Δ_1 (at least for underlying dimensions finite and $\geqslant 5$). The details will not be given. Briefly, a construction analogous to that in step (2) of [9, Sect. 5.4.1] allows us to assume $\nu \geqslant 2$, and utilizing an appropriate $E_{i,\omega}$ transformation (Eichler [5]) we get by with $\nu_1 \geqslant 2$, in 2.3. We can assume $\nu_1 = 2$. An argument as in 2.6(1) and Case (2) of 2.7 shows Λ and Λ^{-1} carry projective transvections to projective shearings. It is now an easy consequence of the assumption of finite dimensions that Λ preserves at least one projective transvection. In light of these facts, the argument of 2.4 works (in essence) for ν, $\nu_1 \geqslant 2$, thus projective transvections are preserved by Λ. Finally, the first paragraph of Section 2C requires only that the underlying Witt indices be $\geqslant 2$.

3. The Nonisomorphisms between Linear and Either Unitary or Symplectic Groups

First let us recall some concepts about linear spaces developed in [11]. For an abstract vector space V over D we have the contragredient map $^\vee\colon \Gamma L(V) \to \Gamma L(V')$ defined by $\check{k}\rho = \mu\rho k^{-1}$ ($k \in \Gamma L(V)$ with associated automorphism μ, $\rho \in V'$) and the naturally induced projective contragredient map $^\vee\colon P\Gamma L(V) \to P\Gamma L(V')$. A subspace W of V' is said to be total if $W^{\top} = 0$. Now fix a total subspace W of V. The set $\{\tilde{x}\,|_W\colon x \in V\}$ forms a total subspace of W', denoted $\tilde{V}$.

3.1. *Let U be any subspace of V. Then the codimension of $U^{\perp} \cap W$ in W is finite if and only if U is finite dimensional, in which case the codimension in question is equal to* $\dim U$ *and* $(U^{\perp} \cap W)^{\top} = U$.

The set of lines in V is denoted by $\mathscr{L}$ and the set of hyperplanes of V determined by the linear functionals in W is denoted by $\mathscr{H}$.

3.2. *Suppose U is a finite dimensional subspace of V and suppose L_1, L_2 are lines in V not contained in U. Then there exists a hyperplane H in $\mathscr{H}$ such that $U \subseteq H$ but $L_1 \nsubseteq H$ and $L_2 \nsubseteq H$. In particular, U is the intersection of all hyperplanes in $\mathscr{H}$ containing U.*

Proof. If $L_2 \nsubseteq L_1 + U$, then by 3.1 we can find linear functionals ρ_1, ρ_2 in W such that $\rho_1(U + L_2) = 0$, $\rho_1(L_1) \neq 0$ and $\rho_2(U + L_1) = 0$, $\rho_2(L_2) \neq 0$. Then take the hyperplane $H \in \mathscr{H}$ determined by $\rho_1 + \rho_2$. If however $L_2 \subseteq$

$L_1 + U$, any hyperplane $H \in \mathscr{H}$ containing U but excluding L_1 must exclude L_2 also. Apply 3.1. Q.E.D.

A subgroup Δ of $P\Gamma L(V)$ is said to be full of projective transvections if $\dim V \geqslant 2$ and

(1) The spaces $L \subseteq H$ of each nontrivial projective transvection in Δ satisfy $L \in \mathscr{L}$ and $H \in \mathscr{H}$.

(2) Given $L \in \mathscr{L}$ and $H \in \mathscr{H}$ with $L \subseteq H$, there is at least one nontrivial projective transvection in Δ with spaces $L \subseteq H$.

For $k \in \Gamma L(V)$ we say that $\hat{k}$ is defined whenever $\check{k}W = W$ and then define $\hat{k} = \check{k}|_W$. The set of elements of $\Gamma L(V)$ for which $\hat{}$ is defined form a subgroup X, say of $\Gamma L(V)$ and $\hat{}$ is an injective homomorphism of X into $\Gamma L(W)$.

3.3. *If σ is any element of $GL(V)$ for which $\hat{\sigma}$ is defined, then*

(1) *the fixed space of $\hat{\sigma}$ is $R^{\perp} \cap W$;*

(2) $\operatorname{res} \sigma < \infty \Leftrightarrow \operatorname{res} \hat{\sigma} < \infty$;

(3) *If* $\operatorname{res} \sigma < \infty$, *then* $\operatorname{res} \sigma = \operatorname{res} \hat{\sigma}$, *and the residual space of $\hat{\sigma}$ is $P^{\perp}$, and $(R^{\perp} \cap W)^{\mathsf{T}} = R$.*

We can well define an induced map $\hat{}: PX \to P\Gamma L(W)$ via $\hat{\bar{k}} = \bar{\hat{k}}$, which is also an injective homomorphism.

3.4. (1) *$\hat{}$ is defined for all elements of Δ.*

(2) *The subgroup $\hat{\Delta}$ of $P\Gamma L(W)$ is full of projective transvections relative to $\tilde{V}$.*

Now we introduce a vector space V_3 and suppose W is a total subspace of V'_3. Suppose $\Delta_3 \subseteq P\Gamma L(V_3)$ is full of projective transvections relative to W. We also return to our general assumption that V is a non-zero regular alternating or trace-valued isotropic skew-hermitian space and $\Delta \subseteq P\Gamma I(V)$ has enough projective transvections. Suppose $\Lambda: \Delta \to \Delta_3$ is an isomorphism. *Our goal is to show that if $\nu \geqslant 3$ and* $\dim V_3 \geqslant 5$ *then Λ does not exist.*

The following preliminary result is proved by a straightforward adaptation of the proof of [10, Sect. 6.1.4] (making appropriate use of 2.1).

3.5. *If τ is a projective transvection in Δ and $\Lambda\tau$ has a representative k_1 in $GL(V_3)$ of residue* 1 *then k_1 is a transvection.*

3.6. *Suppose $\nu \geqslant 3$ and* $\dim V_3 \geqslant 5$. *Let τ be a transvection in $I(V)$ that is projectively in Δ and let k_1 be an element of $\Gamma L(V_3)$ with $\Lambda\bar{\tau} = \bar{k}_1$. Then $\bar{k}_1$ is a projective transvection.*

Proof. The proof is analogous to that of 2.3. Since k_1 is not a radiation, it moves a line in V_3. Based on any such line K_1 we construct a situation as follows. By 3.2 there exists a hyperplane H in $\mathscr{H}$ such that $K_1 \subseteq H$ but $k_1K_1 \nsubseteq H$ and $k_1^{-1}K_1 \nsubseteq H$. By fullness there exists a transvection τ_1 in $\Gamma L(V_3)$, projectively in Δ_3, with spaces $K_1 \subseteq H$. Let k be a representative in $\Gamma I(V)$ of $\Lambda^{-1}\bar{\tau}_1$. Put $\sigma = \tau k \tau^{-1} k^{-1}$ and $\sigma_1 = k_1 \tau_1 k_1^{-1} \tau_1^{-1}$. Then it is easy to verify that R_1 is the plane $K_1 + k_1K_1$, R is a line or plane containing the residual line of τ, $R_1 \cap P_1 = 0$ and P is spanned by isotropic lines. Call this situation S_K (or S) and call R_1, P_1 the spaces associated to S. Let $\mathfrak{A}_S$ (or $\mathfrak{A}$) be the set $\{L_1 \colon L_1$ is a line in P_1 such that for each hyperplane H_1 in $\mathscr{H}$ containing $L_1 + R_1$ there exists a projective transvection $\bar{\tau}_3$ in Δ_3 with spaces $L_1 \subseteq H_1$ which has a pullback in $\Gamma I(V)$ which stabilizes R and P and moves an isotropic line in $P\}$.

In step (1) we show that if $\mathfrak{A} \neq \varnothing$ then $k_1|_{P_1}$ is a radiation. In step (2) we show that all situations S with $\mathfrak{A} = \varnothing$ (if such exist) have the same associated space P_1. In step (3), we construct three situations as above with pairwise distinct P_1's and the result follows from steps (1) and (2) and 3.5.

(1) Suppose $\mathfrak{A} \neq \varnothing$.

(1A) Here we show k_1 stabilizes all lines in $\mathfrak{A}$. So fix $L_1 \in \mathfrak{A}$ and let H_1 be a hyperplane in $\mathscr{H}$ containing $L_1 + R_1$. Since $L_1 \in \mathfrak{A}$, there exists a transvection τ_3, projectively in Δ_3, with spaces $L_1 \subseteq H_1$ and an element j in $\Gamma I(V)$ such that $\bar{j} \in \Delta$, $\Lambda\bar{j} = \bar{\tau}_3$ and $jP = P$ while j moves an isotropic line L say, of P. Take a transvection τ_L, projectively in Δ, with residual line L. Put $\sigma_2 = j\tau_L j^{-1}\tau_L^{-1}$ and $\sigma_3 = \tau_3 j_1 \tau_3^{-1} j_1^{-1}$ where j_1 is a representative for $\Lambda\tau_L$ in $\Gamma L(V_3)$. Then it is easy to see σ_2, σ_3 have the following properties: $\sigma_2 \in I(V)$, $\bar{\sigma}_2 \in \Delta$, res $\sigma_2 = 2$, $R_2 \subseteq P$, and $\sigma_3 \in GL(V_3)$, $\bar{\sigma}_3 \in \Delta_1$, $1 \leqslant$ res $\sigma_3 \leqslant 2$, $R_3 \subseteq P_1$, $R_1 \subseteq P_3$. Also $L_1 \subseteq R_3$ or $H_1 = P_3$, and $H_1 \supseteq P_3$ or $L_1 = R_3$.

We now show k_1 stabilizes P_1. Since $R_2 \subseteq P$, by standard methods, k_1 stabilizes R_3 and P_3. Applying Zorn's lemma, there is a nonzero subspace Z_1 of P_1 which is maximal with respect to being stabilized by k_1. If $Z_1 \neq P_1$ it is easily seen that there is a transvection in $GL(V_3)$, projectively in Δ_3, that permutes with σ_1 and that carries R_3 outside Z_1 (but still in P_1). Conjugating σ_3 by this transvection and carrying things back to $\Gamma I(V)$ in the usual way we obtain a new σ'_2 and σ'_3 with $R'_3 \nsubseteq Z_1$. Again we find k_1 stabilizes R'_3, hence k_1 stabilizes $R'_3 + Z_1$ contradicting the maximality of Z_1. Hence $Z_1 = P_1$ and k_1 stabilizes P_1.

Next we show k_1 stabilizes L_1. First suppose k_1 stabilizes all hyperplanes in $\mathscr{H}$ which contain $L_1 + R_1$. Then k_1 stabilizes $L_1 + R_1$ by 3.2, so k_1 stabilizes

$P_1 \cap (L_1 + R_1) = L_1$. Secondly, suppose some hyperplane H_1 in $\mathscr{H}$ containing $L_1 + R_1$ is not stabilized by k_1. Define τ_3, σ_2 and σ_3 as above. Since $H_1 \neq P_3$, we have $L_1 \subseteq R_3$ by the listed properties of σ_3. It is easily seen there is a transvection in $GL(V_3)$, projectively in Δ_3, that permutes with σ_1, and that carries R_3 to a subspace of P_1 which intersects R_3 in L_1. Conjugating σ_3 by this transvection and pulling things back to $\Gamma I(V)$, we obtain a new σ'_2, σ'_3 satisfying the same properties as σ_2, σ_3, and with $R_3 \cap R'_3 = L_1$. So k_1 stabilizes $R'_3 \cap R_3 = L_1$.

(1B) Next we show any plane T in P_1 which contains a line in $\mathfrak{A}$ is spanned by lines in $\mathfrak{A}$. For suppose T is spanned by the lines L_1 and L_3 with $L_1 \in \mathfrak{A}$. Use 3.2 to find a hyperplane H_3 in $\mathscr{H}$ containing $R_1 + L_3$ but not containing L_1. Let τ_3 be a transvection, projectively in Δ_3, with spaces $L_3 \subseteq H_3$. Then $T = L_1 + \tau_3 L_1$. It is not hard to see that as H_5 ranges over the hyperplanes in $\mathscr{H}$ containing $L_1 + R_1$, $\tau_3 H_5$ ranges over the hyperplanes in $\mathscr{H}$ containing $\tau_3 L_1 + R_1$ and from this that $\tau_3 L_1$ is in $\mathfrak{A}$. So T is spanned by lines in $\mathfrak{A}$. In particular, k_1 stabilizes T.

(1C) Finally we show k_1 stabilizes all lines in P_1. Fix a line L_1 in $\mathfrak{A}$. Let L_3 be any line distinct from L_1 in P_1. Choose a plane T in P_1 containing L_1 but not equal to $L_1 + L_3$ (possible since $\dim V_3 \geqslant 5$). By step (1B) we can write $T = L_1 + L_5$ with $L_5 \in \mathfrak{A}$. Then, by step (1B) again, k_1 stabilizes $L_3 + L_1$ and $L_3 + L_5$, and hence k_1 stabilizes their intersection, which is L_3 as required. So $k_1 |_{P_1}$ is a radiation.

(2) Suppose S and S' are two situations with $\mathfrak{A} = \mathfrak{A}' = \varnothing$. We want to show $P_1 = P'_1$. Suppose not. By the same Witt index considerations as in step (2) of 2.3, we can choose distinct isotropic lines L, L' in $P \cap P'$. Choose transvections τ_L, $\tau_{L'}$, projectively in Δ with residual lines L, L'. By standard methods $\Lambda\bar{\tau}_L$ has a representative j_1 in $GL(V_3)$ which is identity on $P_1 + P'_1$, thus by 3.5 and since $P_1 \neq P'_1$, j_1 must be a transvection with fixed hyperplane $P_1 + P'_1$. Similarly $\Lambda\tau_{L'} = \bar{j}'_1$ where j'_1 is a transvection in $GL(V_3)$ with fixed hyperplane $P_1 + P'_1$. We claim j_1 and j'_1 also have the same residual line. Consider j_1. Now j_1 permutes with σ_1, hence j_1 stabilizes R_1. Clearly $R_1 \not\subseteq P_1 + P'_1$, the fixed space of j_1, so by 1.10 R_1 contains the residual line X_1 say, of j_1. Similarly $R'_1 \supseteq X_1$. Thus if $R_1 \neq R'_1$, then $X_1 = R_1 \cap R'_1$ and if $R_1 = R'_1$ it is clear that $X_1 = R_1 \cap (P_1 + P'_1) = R'_1 \cap (P_1 + P'_1)$. And similarly for j'_1, establishing the claim. So τ_L and $\tau_{L'}$ have distinct residual lines whereas j_1 and j'_1 are transvections having the same spaces. Now a standard application of 2.1 yields a contradiction.

(3) Here we construct three situations S, S', S'' with pairwise distinct spaces P_1, P'_1, P''_1. First we show how to construct three situations with distinct 'R's and then use duality to get the desired situations with distinct 'P's.

(3A) Suppose first that for some situation based on the line $K_1 = D_1x_1$ say, k_1 does not stabilize R_1. Define $x'_1 = k_1x_1$ and $x''_1 = k_1x'_1$ and construct situations S', S'' based on the lines $D_1x'_1$, $D_1x''_1$. Then R_1, R'_1, and R''_1 are pairwise distinct.

(3B) Now suppose that for all situations S, k_1 stabilizes the associated plane R_1. Start with any line K_1 in V_1 moved by k_1. By 1.15, k_1 moves a line K'_1 outside $R_1 = K_1 + k_1K_1$ and also moves a line K''_1 outside $K_1 + k_1K_1 + K'_1 + k_1K'_1$. Then situations based on the lines K_1, K'_1, K''_1 have distinct associated planes R_1, R'_1, R''_1, as desired.

(3C) Finally to obtain situations with distinct P_1's, we consider the isomorphism $\hat{} \circ \Lambda: \Delta \to \hat{\Delta}_3$. Here $\hat{\Delta}_3$ is full of projective transvections relative to the total subspace $\tilde{V}$ of W' and dim $W \geqslant 5$. So we can construct situations analogous to the above for the element $\hat{k}_1$ of $\Gamma L(W)$, in particular, by steps (3A) and (3B) we can produce three situations with pairwise distinct associated planes R_1, R'_1, R''_1, *in* W. Then pulling things back from $\Gamma L(W)$ to $\Gamma L(V_3)$ via $\hat{}$, we find that we obtain three situations with associated spaces $P_1 = R^{\top}{}_1$, etc. And P_1, P'_1, P''_1 are distinct since $R_1 = P^{\perp}{}_1$ etc., and the R_1's are distinct. This completes the proof. Q.E.D.

Having established 3.6 we obtain our main result 3.7 by a proof almost identical to that of [10, Sect. 6.1.5] (making appropriate use of 3.2).

3.7. *Suppose* $\nu \geqslant 3$ *and* dim $V_3 \geqslant 5$. *Then* Λ *does not exist.*

A formulation and proof of the nonprojective analog of 3.7 is left to the reader.

Recall from [11, p. 131] that if M_3 is a bounded $\mathfrak{o}_3$-module on V_3 where $\mathfrak{o}_3$ is an integral domain possessing the division ring of quotients D_3, and if $\mathfrak{a}_3$ is any nonzero integral ideal of $\mathfrak{o}_3$, then $GL(M_3; \mathfrak{a}_3)$ is full of transvections relative to the total subspace $D_3M^{\#}{}_3$ of V'_3 if dim $V_3 \geqslant 2$. We also know that in the analogous symplectic or unitary situation, $I(M; \mathfrak{a})$ has enough transvections if the underlying Witt index is $\geqslant 3$. Thus we have the following special cases of 3.7 and its nonprojective analog.

3.8. THEOREM. *Let* $\mathfrak{o}$ *and* $\mathfrak{o}_1$ *be integral domains which possess division rings of quotients* D *and* D_1. *Let* V *be a regular alternating or trace-valued isotropic general hermitian space* (*with form* q) *over* D, *of Witt index* $\geqslant 3$ *and let* M *be a bounded* $\mathfrak{o}$-*module in* V *such that for each isotropic vector* a *in* M, $q(a, M)$ *is contained in a fractional right* (*left*) *ideal with respect to* $\mathfrak{o}$ *if* V *is a right* (*left*) *space. Let* V_1 *be a linear space of dimension* $\geqslant 5$ *over* D_1 *and let* M_1 *be a bounded* $\mathfrak{o}$-*module on* V_1. *Let* $\mathfrak{A}$ *and* $\mathfrak{A}_1$ *be integral ideals of* $\mathfrak{o}$ *and* $\mathfrak{o}_1$. *If* Δ *and* Δ_1 *are groups such that*

$$PI(M; \mathfrak{a}) \subseteq \Delta \subseteq P\Gamma I(V), \qquad \textit{and} \qquad PGL(M_1; \mathfrak{a}_1) \subseteq \Delta_1 \subseteq P\Gamma L(V_1)$$

then Δ is not isomorphic to Δ_1. If Γ and Γ_1 are groups such that

$$I(M; \mathfrak{a}) \subseteq \Gamma \subseteq \Gamma I(V), \qquad \textit{and} \qquad GL(M_1; \mathfrak{a}_1) \subseteq \Gamma_1 \subseteq \Gamma L(V_1)$$

then Γ is not isomorphic to Γ_1.

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